

NON-SELF-ADJOINT JACOBI MATRICES WITH RANK ONE IMAGINARY PART

YURY ARLINSKIĬ AND EDUARD TSEKANOVSKIĬ

ABSTRACT. We develop direct and inverse spectral analysis for finite and semi-infinite non-self-adjoint Jacobi matrices with a rank one imaginary part. It is shown that given a set of n not necessarily distinct non-real numbers in the open upper (lower) half-plane uniquely determines a $n \times n$ Jacobi matrix with a rank one imaginary part having those numbers as its eigenvalues counting multiplicity. An algorithm for reconstruction for such finite Jacobi matrices is presented. A new model complementing the well known Livsic triangular model for bounded linear operators with rank one imaginary part is obtained. It turns out that the model operator is a non-self-adjoint Jacobi matrix and it follows from the fact that any bounded, prime, non-self-adjoint linear operator with rank one imaginary part acting on some finite-dimensional (resp., separable infinite-dimensional Hilbert space) is unitary equivalent to a finite (resp., semi-infinite) non-self-adjoint Jacobi matrix. This result strengthens the classical Stone theorem established for self-adjoint operators with simple spectrum. We establish the non-self-adjoint analogs of the Hochstadt and Gesztesy–Simon uniqueness theorems for finite Jacobi matrices with non-real eigenvalues as well as an extension and refinement of these theorems for finite non-self-adjoint tri-diagonal matrices to the case of mixed eigenvalues, real and non-real. A unique Jacobi matrix, unitarily equivalent to the operator of integration $(\mathcal{F}f)(x) = 2i \int_x^l f(t)dt$ in the Hilbert space $\mathcal{L}_2[0, l]$ is found as well as spectral properties of its perturbations and connections with well known Bernoulli numbers. We also give the analytic characterization of the Weyl functions of dissipative Jacobi matrices with a rank one imaginary part.

CONTENTS

1. Introduction	2
2. Bounded prime dissipative operators and their characteristic functions	7
2.1. Bounded prime dissipative operators	7
2.2. The Livsic characteristic function of non-self-adjoint bounded operators	8
2.3. Triangular models for bounded dissipative operators with rank one imaginary parts	10
2.4. Linear-fractional transformation of the characteristic function	12

2.5.	The Weyl function of a bounded dissipative operator	13
3.	Jacobi matrices and their Weyl functions	18
3.1.	Complex Jacobi matrices and corresponding Jacobi operators	18
3.2.	Finite complex Jacobi matrices	19
3.3.	Semi-infinite Jacobi matrices	21
3.4.	The m -function approach in inverse spectral problems for self-adjoint Jacobi matrices	
3.5.	Mixed given data and uniqueness for finite self-adjoint Jacobi matrixes	25
4.	Dissipative Jacobi matrices with a rank one imaginary part	26
4.1.	Finite and semi-infinite dissipative Jacobi matrices	26
4.2.	The m -functions of a finite dissipative Jacobi matrix with a rank one imaginary part	
5.	Inverse spectral problems for finite Jacobi matrices with a rank one imaginary part	34
5.1.	Reconstruction of a finite dissipative Jacobi matrix from its eigenvalues	34
5.2.	The connection between triangular and Jacobi forms of a prime dissipative operator	
5.3.	A non-self-adjoint analog of the Hochstadt and Gesztesy–Simon theorems	43
5.4.	Extension and refinement of the Hochstadt and Gesztesy–Simon uniqueness theorems	
6.	A non-self-adjoint analog of the Stone theorem	46
7.	Dissipative Volterra operators with rank one imaginary parts and the corresponding semigroups	
	References	57

1. INTRODUCTION

Self-adjoint (or real) finite and semi-infinite Jacobi matrices of the form

$$(1.1) \quad J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & 0 & a_{n-1} & b_n \end{pmatrix}$$

and

$$(1.2) \quad J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & 0 & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where $a_k > 0$, and b_k are real numbers for all $k = 1, 2, \dots$ play an important role in various problems of mathematical analysis and theoretical and mathematical physics. They appear as the discrete analog of Sturm–Liouville operators, in inverse spectral theory, in the study of the classical moment problem, and in the investigation of completely integrable nonlinear lattices [1], [8], [9], [12], [15], [18], [29], [32], [33].

More general tri-diagonal matrices with complex entries (or complex Jacobi matrices) also attracted much attention as a useful tool in the study of orthogonal polynomials, in the theory of continued fractions, and in numerical analysis [6], [7], [34].

Let the linear space \mathbb{C}^n of columns be equipped by the usual inner product

$$(x, y) = \sum_{k=1}^n x_k \overline{y_k}$$

and let $l_2(\mathbb{N})$ be the Hilbert space of square summable complex-valued sequences

$$x = \{x_1, x_2, \dots, x_k, \dots\}$$

which we consider as semi-infinite vector-columns with the inner product given by

$$(x, y) = \sum_{k=1}^{\infty} x_k \overline{y_k}.$$

Let

$$\delta_k := (\underbrace{0, \dots, 0}_k, 1, 0, \dots)^T, \quad k = 1, 2, \dots$$

Then the vectors $\{\delta_k\}$ form an orthonormal basis in \mathbb{C}^n (resp., $l_2(\mathbb{N})$). An $n \times n$ complex Jacobi matrix J determines a linear operator in the Hilbert space \mathbb{C}^n by means of the matrix product $J \cdot x$. For the semi-infinite case we will suppose in addition that

$$(1.3) \quad \sup_k \{|a_k| + |b_k|\} < \infty, \quad k \in \mathbb{N}.$$

This condition is necessary and sufficient [2] for boundedness of the Jacobi operator in $l_2(\mathbb{N})$ defined as $J \cdot x$, where J is a semi-infinite complex Jacobi matrix. Moreover, under the condition $a_k \neq 0$ for all k , the vector δ_1 is cyclic for the Jacobi operator J . The classical Stone theorem [1], [2], [32] states that every self-adjoint operator with simple spectrum in a separable Hilbert space \mathcal{H} is unitarily equivalent to the operator determined by a self-adjoint Jacobi matrix.

In this paper we consider finite and semi-infinite tri-diagonal matrices of the form (1.1) and (1.2) with

$$(1.4) \quad \operatorname{Im} b_1 > 0, \quad b_k = \bar{b}_k \quad (k = 2, 3, \dots), \quad a_k > 0 \quad (k \in \mathbb{N}).$$

Such matrices determine bounded Jacobi operators J in \mathbb{C}^n or in $l_2(\mathbb{N})$ that posses the properties

$$\begin{aligned} \operatorname{Im} (Jx, x) &\geq 0 \quad \text{for all } x \in \mathbb{C}^n \text{ or } x \in l_2(\mathbb{N}), \\ \operatorname{ran} (J - J^*) &= \{\lambda \delta_1, \lambda \in \mathbb{C}\}. \end{aligned}$$

In the following we will call such tri-diagonal matrices *the dissipative Jacobi matrices with rank one imaginary part*. Because the vector δ_1 is cyclic for J , the dissipative operator J is prime [10], [11], [27], and therefore it has no real eigenvalues.

M. S. Livsic [27] (see also [11]) constructed a triangular model for a bounded prime dissipative operator A with a rank one imaginary part. The method of construction is based on the factorization of the characteristic function $W(z)$ of the operator A . The model operator is in general a coupling of two triangular operators. The first one is given by a finite or semi-infinite triangular matrix of the form

$$\dot{A} = \begin{pmatrix} \alpha_1 + \frac{i}{2}\beta_1^2 & i\beta_1\beta_2 & \cdot & \cdot & \cdot & i\beta_1\beta_n \\ 0 & \alpha_2 + \frac{i}{2}\beta_2^2 & \cdot & \cdot & \cdot & i\beta_2\beta_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \alpha_n + \frac{i}{2}\beta_n^2 \end{pmatrix}$$

or

$$\dot{A} = \begin{pmatrix} \alpha_1 + \frac{i}{2}\beta_1^2 & i\beta_1\beta_2 & \cdot & \cdot & \cdot & i\beta_1\beta_k & \cdot & \cdot & \cdot \\ 0 & \alpha_2 + \frac{i}{2}\beta_2^2 & \cdot & \cdot & \cdot & i\beta_2\beta_k & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \alpha_k + \frac{i}{2}\beta_k^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where $\{\alpha_k\}_{k=1}^n$ are real numbers and $\{\beta_k\}_{k=1}^n$ are positive numbers and $z_k = \alpha_k + i(\beta_k^2/2)$, $k = 1, 2, \dots$ are the non-real eigenvalues of \dot{A} . The second one is the operator in the Hilbert space $\mathcal{L}_2[0, l]$ given by

$$(\dot{B}f)(x) = \alpha(x)f(x) + i \int_x^l f(t)dt,$$

where α is a bounded nondecreasing right continuous function on $[0, l]$. The operator \dot{B} has pure real spectrum which coincides with the closure of the set $\{\alpha(x), x \in [0, l]\}$.

One of the main results of the paper states that *a bounded prime dissipative operator with a rank one imaginary part acting on a separable Hilbert space is unitarily equivalent to the operator determined by a dissipative Jacobi matrix with rank one imaginary part. This result is a non-self-adjoint analog of the classical Stone theorem, established for self-adjoint operators with simple spectrum in [1], [2], [32]. Thus, dissipative Jacobi matrices with rank one imaginary parts provide new models for the prime bounded linear operators with rank one imaginary parts. The entries of the corresponding Jacobi matrix can be found*

using the continued fraction (J-fraction) expansion [34]

$$M(z) = \frac{-1}{z - b_1} + \frac{-a_1^2}{z - b_2} + \frac{-a_2^2}{z - b_3} + \dots + \frac{-a_{n-1}^2}{z - b_n} + \dots,$$

where $W(z)$ is the characteristic function, $M(z) = \frac{i}{\beta}(W(z) - 1)$, and $\beta = \lim_{z \rightarrow \infty} (iz(1 - W(z)))$.

By the Livsic theorem [27], a prime dissipative Volterra operator A acting on the Hilbert space \mathcal{H} with rank one imaginary part such that

$$(A - A^*)h = 2i l(h, e)e$$

for all $h \in \mathcal{H}$, $\|e\| = 1$, is unitarily equivalent to the integration operator of the form

$$(Ff)(x) = 2i \int_x^l f(t)dt$$

acting on the Hilbert space $\mathcal{L}_2[0, l]$. *In this paper a unique Jacobi matrix unitarily equivalent to the integration operator \mathcal{F} is found, and some connections with the well known Bernoulli numbers are established.*

We also show that for n not necessarily distinct given complex numbers in the open upper half-plane, there exists a unique $n \times n$ Jacobi matrix having those numbers as its eigenvalues counting multiplicity. Two algorithms for the reconstruction of a finite Jacobi matrix of the form (1.1) from its eigenvalues are presented. We obtain the following non-self-adjoint analogs of the Hochstadt and Gesztesy–Simon uniqueness theorems for self-adjoint finite Jacobi matrices: Let J be an $n \times n$ Jacobi matrix satisfying conditions (1.4). Suppose that the eigenvalues z_1, \dots, z_k (taken from the open upper half-plane) with their multiplicities l_1, \dots, l_k are known together with $b_1, a_1, b_2, \dots, a_{n-r-1}, b_{n-r}$, where $r = l_1 + l_2 + \dots + l_k$. Then $a_{n-r}, b_{n-r+1}, \dots, a_{n-1}, b_n$ are uniquely determined. This as well as some extensions and refinements of the Hochstadt and Gesztesy–Simon uniqueness theorems concerning the case of dissipative but non prime tri-diagonal matrices with rank one imaginary part are established.

Our main tools are the Livsic characteristic function, its linear-fractional transformation [10], [11], [27] [28], and the Weyl function of a Jacobi matrix. Moreover, in order to establish the above mentioned non-self-adjoint analogs of the Hochstadt [19] and Gesztesy–Simon [15] uniqueness theorems in the theory of inverse spectral problems we develop further the approach Gesztesy–Simon used in [15].

Our paper is organized as follows: In Section 2 we present some properties of prime, bounded, dissipative operators with a rank one imaginary part and the Livsic theory of their characteristic functions and triangular models. We define the Weyl function of such operators and give its analytic characterizations. We emphasize that the important role of the Weyl function (Weyl–Titchmarsh function) in the spectral theory of self-adjoint differential and difference operators and its applications to nonlinear equations is well-known (see [1], [5], [8], [9], [15], [16], [33]). In Section 3 some basic properties of complex Jacobi matrices and the corresponding Jacobi operators as well as a survey of the inverse spectral problems for finite and semi-infinite self-adjoint Jacobi matrices are given. We begin our study of dissipative Jacobi matrices with a rank one imaginary part in Section 4. In particular, the connection between m_+ - and m_- -functions of a finite dissipative Jacobi matrix is established. In Section 5 we present solutions of the inverse spectral problems for finite dissipative Jacobi matrices. The non-self-adjoint analogs of the Hochstadt and Gesztesy–Simon theorems are presented in Subsections 5.3 and 5.4. The non-self-adjoint analog of Stone’s theorem is considered in Section 6. *We also obtain in Section 6 that if the Livsic characteristic function $W(z)$ of a bounded dissipative operator with rank one imaginary part satisfies the condition $W(-z) = W^{-1}(z)$ in some neighborhood of infinity, then the corresponding Jacobi matrix possesses the property $\operatorname{Re} b_1 = b_2 = \dots = 0$. In Section 7 we find the Jacobi matrix corresponding to the integration operator and establish that the perturbation of its upper left entry b_1 leads to a Jacobi matrix with a complete system of eigensubspaces.*

We will use the following notation: For a bounded linear operator A in \mathcal{H} , A_R and A_I denote its Hermitian components

$$A_R = \frac{A + A^*}{2}, \quad A_I = \frac{A - A^*}{2i},$$

and $\rho(A)$ and $\sigma(A)$ denote the resolvent set and the spectrum of A . By \mathbb{C}_+ (resp., \mathbb{C}_-) we denote the open upper (resp., lower) half-plane. For a continued fraction

$$\cfrac{c_1}{d_1 + \cfrac{c_2}{d_2 + \cfrac{c_3}{d_3 + \dots}}}$$

we use the notation (see [23], [24], [34])

$$\cfrac{c_1}{d_1} + \cfrac{c_2}{d_2} + \cfrac{c_3}{d_3} + \dots$$

Finally, we would like to mention that the fundamental research provided by Yuri Berezanskii, Harry Hochstadt, Fritz Gesztesy and Barry Simon in the theory of self-adjoint Jacobi matrices as well as by Moshe Livsic in the theory of non-self-adjoint operators inspired and encouraged us to make a new step and to develop the direct and inverse spectral analysis of finite and semi-infinite non-self-adjoint Jacobi and tri-diagonal matrices with a rank one imaginary part as presented in this paper.

2. BOUNDED PRIME DISSIPATIVE OPERATORS AND THEIR CHARACTERISTIC FUNCTIONS

2.1. Bounded prime dissipative operators. Let \mathcal{H} be a complex separable Hilbert space with the inner product (\cdot, \cdot) and the norm $\|\cdot\|$. The operator A is called *dissipative* if $A_I \geq 0$. The resolvent set $\rho(A)$ of dissipative operator A contains the open lower half-plane and the estimate

$$\|(A - zI)^{-1}\| \leq \frac{1}{|\operatorname{Im} z|}, \operatorname{Im} z < 0$$

holds. Let

$$(2.1) \quad \mathcal{H}_s = \overline{\operatorname{span} \{A^n A_I \mathcal{H}, n = 0, 1, \dots\}}.$$

The subspace \mathcal{H}_s reduces A and $A|_{(\mathcal{H} \ominus \mathcal{H}_s)}$ is a self-adjoint operator. The operator $A|_{\mathcal{H}_s}$ is called the *prime part* of A .

Definition 2.1. *A bounded linear operator A in a separable Hilbert space is called prime or completely non-self-adjoint if there is no reducing invariant subspace on which the operator A is self-adjoint.*

It is well known [10] that an operator A is prime in \mathcal{H} if and only if

$$(2.2) \quad \overline{\operatorname{span} \{A^n A_I \mathcal{H}, n = 0, 1, \dots\}} = \mathcal{H}.$$

Proposition 2.2. *Let A be a dissipative operator with a rank one imaginary part and let g be a vector in \mathcal{H} such that*

$$(2.3) \quad 2A_I h = (h, g)g, \quad h \in \mathcal{H}.$$

Then A is prime if and only if the vector g is cyclic for the real part A_R .

Proof. Suppose that A is prime. Then (2.2) holds. Let us prove that g is a cyclic vector for A_R . Let

$$\mathcal{H}' = \overline{\operatorname{span} \{(A_R)^n g, n = 0, 1, \dots\}} \neq \mathcal{H}.$$

Then \mathcal{H}' and $\mathcal{H}'' = \mathcal{H} \ominus \mathcal{H}'$ are invariant with respect to A_R . Since $\mathcal{H}'' \subset \text{Ker } A_I$, it follows that

$$A_R \upharpoonright \mathcal{H}'' = A \upharpoonright \mathcal{H}'' = A^* \upharpoonright \mathcal{H}'',$$

$A^{*n} \mathcal{H}'' \subset \mathcal{H}$ and $A_I A^{*n} \upharpoonright \mathcal{H}'' = 0$ for all $n = 0, 1, \dots$. Now from (2.2) we obtain that $\mathcal{H}'' = \{0\}$, i.e. g is a cyclic vector for A_R .

Conversely, suppose that the vector g is cyclic for A_R , i.e. $\mathcal{H}' = \mathcal{H}$. Let the subspace \mathcal{H}_s be defined by (2.1). Then

$$A \upharpoonright (\mathcal{H} \ominus \mathcal{H}_s) = A_R \upharpoonright (\mathcal{H} \ominus \mathcal{H}_s)$$

and $\mathcal{H} \ominus \mathcal{H}_s$ as well as \mathcal{H}_s reduces A_R . Because $g \in \mathcal{H}_s$, we get that $\mathcal{H}' \subset \mathcal{H}_s$. It follows that $\mathcal{H}_s = \mathcal{H}$, i.e. A is a prime operator. \square

A prime dissipative operator has no real eigenvalues and its non-real eigenvalues belong to the open upper half-plane. It is known (see [10]) that the non-real spectrum of an operator with compact imaginary part consists of eigenvalues of finite algebraic multiplicities (dimensions of the corresponding root subspaces) and the limit points of non-real spectrum belong to the spectrum of the real part of operator. *Note that any eigensubspace of an operator with a rank one imaginary part which corresponds to a non-real eigenvalue has a dimension equal to one.* Observe also that if $\dim \mathcal{H} = n$ and A is a dissipative operator with a rank one imaginary part A_I , and if (2.3) holds, then as it follows from Definition 2.2 and Theorem 2.6 the following conditions are equivalent:

- (1) A is a prime operator;
- (2) A has no real eigenvalues;
- (3) the vectors $g, Ag, \dots, A^{n-1}g$ are linearly independent.

The following completeness criterion has been established by M. S. Livsic [27].

Theorem 2.3. [27] . *Let A be a bounded prime dissipative operator in a separable Hilbert space \mathcal{H} with the finite trace $\text{Sp} A_I$. Then the closure of the linear span of all root subspaces coincides with \mathcal{H} if and only if*

$$(2.4) \quad \sum_n \text{Im } z_n = \text{Sp} A_I,$$

where $\{z_n\}$ is the set of all non-real eigenvalues of A counting multiplicity.

2.2. The Livsic characteristic function of non-self-adjoint bounded operators. In this subsection we present basic facts of the Livsic characteristic functions theory [10], [11], [27], [28]. We restrict ourselves just to the case of a bounded dissipative operator A with a rank one imaginary part A_I .

A vector g satisfying (2.3) is called a channel vector for A . The function

$$(2.5) \quad W(z) = 1 - i \left((A - zI)^{-1} g, g \right), \quad z \in \rho(A),$$

is called a *characteristic function of A* . Because an operator A is bounded and dissipative the characteristic function $W(z)$ possesses the following properties:

- (1) $|W(z)| \geq 1$ for $z \in \rho(A)$, $\text{Im } z > 0$,
- (2) $|W(z)| \leq 1$, $\text{Im } z < 0$,
- (3) $|W(z)| = 1$ for $z \in \rho(A)$, $\text{Im } z = 0$,
- (4) $W(z) = 1 + i \sum_{n=0}^{\infty} \frac{(A^n g, g)}{z^{n+1}}$ in a neighborhood of infinity.

In addition the function

$$(2.6) \quad G(z, \xi) = \frac{1 - W(z)\overline{W(\xi)}}{i(z - \bar{\xi})} \quad \text{is a nonnegative kernel,}$$

i.e.

$$\sum_{i,j=1}^n \lambda_i \bar{\lambda}_j G(z_j, z_i) \geq 0, \quad (n = 1, \dots)$$

for any set of points $\{z_i\}_{i=1}^n \subset \rho(A)$ and any vector $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$.

Theorem 2.4. [27]. *Let A_1 and A_2 be bounded dissipative operators with a rank one imaginary part. If the corresponding characteristic functions $W_1(z)$ and $W_2(z)$ coincide in some neighborhood of infinity, then the prime parts of A_1 and A_2 are unitarily equivalent.*

Theorem 2.5. [10]. *Let a function $W(z)$ be holomorphic outside some closed bounded subset \mathcal{D} of the closed upper half-plane and possesses the properties*

- (i) $|W(z)| \leq 1$, $\text{Im } z < 0$,
- (ii) $|W(z)| = 1$ for $z \in \mathbb{C} \setminus \mathcal{D}$, $\text{Im } z = 0$,
- (iii) $\lim_{z \rightarrow \infty} W(z) = 1$.

Then there exists a prime bounded dissipative operator with a rank one imaginary part whose characteristic function coincides with $W(z)$.

Theorem 2.6. [27]. *The characteristic function $W(z)$ of a bounded dissipative operator A with a rank one imaginary part admits the following multiplicative representation:*

$$(2.7) \quad W(z) = \prod_{k=1}^N \left(\frac{z - \bar{z}_k}{z - z_k} \right)^{n_k} \cdot \exp \left(-i \int_0^l \frac{dx}{\alpha(x) - z} \right),$$

where $\{z_k\}_{k=1}^N$ ($N \leq \infty$) are the distinct non-real eigenvalues of A , $\{n_k\}$ are their algebraic multiplicities, $\alpha(x)$ is a bounded nondecreasing and continuous from the right function on the interval $[0, l]$.

Observe, that for any operator A and his channel vector g (2.3)

$$\sum_{k=1}^N n_k \operatorname{Im} z_k \leq \frac{\|g\|^2}{2}.$$

Moreover, from Theorems 2.3 and 2.6 it follows that the closure of the linear span of all root subspaces of operator A coincides with \mathcal{H} if and only if

$$\sum_{k=1}^N n_k \operatorname{Im} z_k = \frac{\|g\|^2}{2}.$$

2.3. Triangular models for bounded dissipative operators with rank one imaginary parts. The next theorem provides a triangular model for a finite-dimensional prime dissipative operator with one dimensional imaginary part.

Theorem 2.7. [17], [27]. *Let $\dim \mathcal{H} = n$ and let A be a dissipative operator in \mathcal{H} with a rank one imaginary part A_I . Then there exists an orthonormal basis in \mathcal{H} in which the matrix of the operator A takes the form*

$$\dot{A} = \begin{pmatrix} \alpha_1 + \frac{i}{2}\beta_1^2 & i\beta_1\beta_2 & \cdot & \cdot & \cdot & i\beta_1\beta_n \\ 0 & \alpha_2 + \frac{i}{2}\beta_2^2 & \cdot & \cdot & \cdot & i\beta_2\beta_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \alpha_n + \frac{i}{2}\beta_n^2 \end{pmatrix},$$

where $\{\alpha_k\}_{k=1}^n$ are real numbers and $\{\beta_k\}_{k=1}^n$ are nonnegative numbers. The operator A is prime if and only if the numbers $\{\beta_k\}$ are positive. The characteristic function of the operator A takes the form

$$W(z) = \prod_{\beta_k \neq 0} \frac{z - \alpha_k + i\beta_k^2/2}{z - \alpha_k - i\beta_k^2/2}.$$

Now consider the case of infinite-dimensional operator.

1) Suppose that a bounded prime dissipative operator A with rank one imaginary part has a complete system of root subspaces. Then the characteristic function of A takes the form

$$\prod_{k=1}^{\infty} \frac{z - \bar{z}_k}{z - z_k},$$

where $\{z_k\}$ are the set of not necessarily distinct complex numbers counting multiplicity and such that

- (1) $\text{Im } z_k > 0$,
- (2) $\sum_{k=1}^{\infty} \text{Im } z_k < \infty$,
- (3) $|z_k| < C$.

From the Livsic approach [27] such an operator A is unitarily equivalent to the operator \dot{A} in the Hilbert space $l_2(\mathbb{N})$ given by the triangular matrix

$$\dot{A} = \begin{pmatrix} \alpha_1 + \frac{i}{2}\beta_1^2 & i\beta_1\beta_2 & \cdot & \cdot & \cdot & i\beta_1\beta_k & \cdot & \cdot & \cdot \\ 0 & \alpha_2 + \frac{i}{2}\beta_2^2 & \cdot & \cdot & \cdot & i\beta_2\beta_k & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \alpha_k + \frac{i}{2}\beta_k^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where $\{\alpha_k\}_{k=1}^n$ are real numbers and $\{\beta_k\}_{k=1}^n$ are positive numbers.

2) Suppose that A has only real spectrum. Then the characteristic function of A takes the form

$$W(z) = \exp \left(-i \int_0^l \frac{dx}{\alpha(x) - z} \right).$$

Consider the following operator \dot{B} in the Hilbert space $L_2[0, l]$:

$$(\dot{B}f)(x) = \alpha(x)f(x) + i \int_x^l f(t)dt.$$

The operator A is unitarily equivalent to the prime part of the operator \dot{B} [27].

3) Let A be an arbitrary bounded prime dissipative operator with a rank one imaginary part acting on infinite-dimensional Hilbert space \mathcal{H} . Then its characteristic function admits multiplicative representation (2.7). Consider Hilbert space $\mathfrak{H} = \mathbb{C}^n \oplus L_2[0, l]$ in case when the non-real spectrum of A is finite and consists of n complex numbers counting their algebraic multiplicity and the Hilbert space $\mathfrak{H} = l_2(\mathbb{N}) \oplus L_2[0, l]$ if the non-real spectrum of A is infinite. Let \dot{G} be an operator in \mathfrak{H} given by the block operator matrix

$$\dot{G} = \begin{pmatrix} \dot{A} & \Gamma \\ 0 & \dot{B} \end{pmatrix},$$

where $\Gamma : L_2[0, l] \rightarrow \mathbb{C}^n$ or $\Gamma : L_2[0, l] \rightarrow l_2(\mathbb{N})$ is given by the formula

$$(\Gamma f)(x) = i \left(\sqrt{2 \sum_k \text{Im } z_k} \int_0^l f(t)dt \right) \delta_1,$$

where $\delta_1 \in \mathbb{C}^n$ or $\delta_1 \in l_2(\mathbb{N})$ is the vector with the first component equal to 1 and remaining components equal to zero. Then A is unitarily equivalent to the prime part of \hat{G} [27].

2.4. Linear-fractional transformation of the characteristic function. Let T be a bounded linear operator acting on some Hilbert space \mathcal{H} and let h be a nonzero vector in \mathcal{H} . Consider the family of rank one perturbations of T given by

$$T_t = T + t(\cdot, h)h,$$

where t is a complex number. One can easily derive the following relation

$$(2.8) \quad ((T_t - zI)^{-1}h, h) = \frac{((T - zI)^{-1}h, h)}{1 + t((T - zI)^{-1}h, h)}, \quad z \in \rho(T_t) \cap \rho(T).$$

Moreover, the number $z_0 \in \rho(T)$ is the eigenvalue of T_t if and only if z_0 satisfies the equation

$$1 + t((T - z_0I)^{-1}h, h) = 0.$$

Let A be a dissipative operator in \mathcal{H} with a rank one imaginary part A_I and let $g \in \mathcal{H}$ such that (2.3) holds. Define the following function

$$(2.9) \quad V(z) = \frac{1}{2}((A_R - zI)^{-1}g, g), \quad z \in \rho(A_R).$$

The function $V(z)$ is holomorphic on the domain $C \setminus [a, b]$, where a and b are the lower and upper bounds of the spectrum of self-adjoint operator A_R , and possesses the properties

- (1) $\frac{V(z) - V^*(z)}{z - \bar{z}} \geq 0, \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-,$
- (2) $V^*(\bar{z}) = V(z),$
- (3) $V(z) = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{((A_R)^n g, g)}{z^{n+1}}$ in a neighborhood of infinity.

The properties (1) and (2) mean that $V(z)$ is a Herglotz–Nevanlinna function and admits the integral representation

$$V(z) = \int_a^b \frac{d\Omega(t)}{t - z},$$

where $\Omega(t) = (E(t)g, g)/2$ and $E(t)$ is a resolution of identity for the operator A_R . Since

$$A_R = A - \frac{i}{2}(\cdot, g)g,$$

from (2.8) and (2.5) it follows that ([10])

$$(2.10) \quad \begin{cases} V(z) = i \frac{W(z) - 1}{W(z) + 1} \\ W(z) = \frac{1 - iV(z)}{1 + iV(z)} \end{cases}, \quad z \in \rho(A) \cap \rho(A_R).$$

The non-real spectrum of A consists of all z , $\text{Im } z > 0$ which are solutions of the equation

$$V(z) = i.$$

2.5. The Weyl function of a bounded dissipative operator. In this subsection we define and study the *Weyl function* of a prime dissipative operator with rank one imaginary part. In the classical case of the Weyl functions of self-adjoint differential and difference operators we refer to [1], [5], [8], [9], [15], [16], [33].

Definition 2.8. *Let A be a prime dissipative operator with a rank one imaginary part. Let $e \in \text{ran}(A - A^*)$, $\|e\| = 1$. The functions*

$$(2.11) \quad m_A(z) = ((A - zI)^{-1}e, e), \quad z \in \rho(A)$$

and

$$(2.12) \quad m_{A_R}(z) = ((A_R - zI)^{-1}e, e), \quad z \in \rho(A_R)$$

are said to be the *Weyl functions* of the operator A and A_R , correspondingly.

Since $e \in \text{ran}(A - A^*)$ and $\|e\| = 1$, the operator A_I takes the form

$$A_I f = l(f, e)e,$$

where $l > 0$. Observe that by Proposition 2.2 the vector e is cyclic for A and A_R . The function $m_A(z)$ is holomorphic on the resolvent set of A which includes the open lower half-plane and the neighborhood of infinity of the form $|z| > \|A\|$ and in this neighborhood

$$m_A(z) = - \sum_{k=0}^{\infty} \frac{(A^k e, e)}{z^{k+1}}.$$

The function $m_{A_R}(z)$ is a Herglotz-Nevanlinna function holomorphic on the resolvent set of A_R and for $|z| > \|A_R\|$

$$m_{A_R}(z) = - \sum_{k=0}^{\infty} \frac{((A_R)^k e, e)}{z^{k+1}}.$$

Relations (2.3) and (2.9) yield

$$m_{A_R}(z) = \frac{1}{l} V(z), \quad z \in \rho(A_R).$$

From (2.3) and (2.5) it follows that the characteristic function $W(z)$ of A and the Weyl function $m_A(z)$ are connected via relations

$$(2.13) \quad W(z) = 1 - 2il m_A(z), \quad m_A(z) = \frac{i}{2l}(W(z) - 1), \quad z \in \rho(A).$$

The relation (2.8) gives the following connections between the Weyl functions of A and A_R

$$(2.14) \quad \begin{aligned} m_A(z) &= \frac{i m_{A_R}(z)}{i - l m_{A_R}(z)}, \\ m_{A_R}(z) &= \frac{i m_A(z)}{i + l m_A(z)}, \quad z \in \rho(A) \cap \rho(A_R). \end{aligned}$$

From (2.14) it follows that the non-real eigenvalues of A coincide with poles of the function $m_A(z)$ and zeroes of the function $i - l m_{A_R}(z)$. The following theorem directly follows from Theorems 2.4 and relations (2.13).

Theorem 2.9. *Let A_1 and A_2 be two prime dissipative operators with rank one imaginary parts. If the Weyl functions of A_1 and A_2 coincide in some neighborhood of infinity, then the operators A_1 and A_2 are unitarily equivalent.*

Suppose that a Hilbert space \mathcal{H} is decomposed as $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Then every bounded operator A in \mathcal{H} has the block-operator matrix representation with respect to this decomposition

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

According to the Schur-Frobenius formula the relation

$$(2.15) \quad P_{\mathcal{H}_1}(A - zI)^{-1}|_{\mathcal{H}_1} = (-zI + A_{11} - A_{12}(A_{22} - zI)^{-1}A_{21})^{-1}$$

holds for $z \in \rho(A) \cap \rho(A_{22})$. Here $P_{\mathcal{H}_1}$ is the orthogonal projection in \mathcal{H} onto \mathcal{H}_1 . Let A be a dissipative operator with a rank one imaginary part and let $\mathcal{H}_1 = \text{ran}(A - A^*)$. Then $\mathcal{H}_2 = \text{Ker}(A - A^*)$, the operator A_{22} is self-adjoint in \mathcal{H}_2 , and $A_{12} = A_{21}^*$. It follows that

$$A_{11}e = be, \quad A_{12}f = a(f, h)e, \quad f \in \mathcal{H}_2, \quad A_{21}e = \bar{a}h,$$

where $e \in \text{ran}(A - A^*)$ and $h \in \text{Ker}(A - A^*)$ are the unit vectors and $\text{Im } b > 0$. Hence (2.15) takes the form for $z \in \rho(A) \cap \rho(A_{22})$

$$(2.16) \quad ((A - zI)^{-1}e, e) = \frac{1}{-z + b - |a|^2 ((A_{22} - zI)^{-1}h, h)}.$$

If A is a prime operator, then $a \neq 0$ and in this case for $z \in \rho(A) \cap \rho(A_{22})$

$$(2.17) \quad ((A_{22} - zI)^{-1}h, h) = \frac{1}{|a|^2} \left(-z + b - \frac{1}{((A - zI)^{-1}e, e)} \right).$$

It follows that

$$\begin{aligned} ((A_{22} - zI)^{-1}h, h) &= \frac{1}{|a|^2} \left(b - z - \frac{1}{m_A(z)} \right), \quad z \in \rho(A) \cap \rho(A_{22}), \\ ((A_{22} - zI)^{-1}h, h) &= \frac{1}{|a|^2} \left(\operatorname{Re} b - z - \frac{1}{m_{A_R}(z)} \right), \quad z \in \rho(A_R) \cap \rho(A_{22}). \end{aligned}$$

In addition, the number z_0 , $\operatorname{Im} z_0 > 0$ is the eigenvalue of A if and only if z_0 satisfies the equation

$$b - z - |a|^2 ((A_{22} - zI)^{-1}h, h) = 0.$$

The corresponding eigenspace is

$$\operatorname{span} \{ e - \bar{a}(A_{22} - z_0 I)^{-1}h \}.$$

The next theorem gives the analytical characterization of the Weyl functions of prime dissipative operators with a rank one imaginary part.

Theorem 2.10. *Let $M(z)$ be a function holomorphic outside some closed bounded subset \mathcal{D} of the closed upper half-plane. The following statements are equivalent:*

- (i) *The function $M(z)$ is the Weyl function of some bounded prime dissipative operator with a rank one imaginary part.*
- (ii) (a) *The function $M(z)$ has the asymptotic expansion at infinity*

$$(2.18) \quad M(z) = -\frac{1}{z} - \frac{b}{z^2} + O\left(\frac{1}{z^3}\right),$$

where $\operatorname{Im} b > 0$.

- (b) *The function*

$$W(z) = 1 - 2i \operatorname{Im} b M(z)$$

is the characteristic function of some dissipative bounded operator with a rank one imaginary part.

- (iii) (a) *The function $M(z)$ has the expansion (2.18) at infinity.*
- (b) *The function*

$$(2.19) \quad \mathcal{K}(z, \xi) := \frac{M(z) - \overline{M(\xi)} + 2i \operatorname{Im} b M(z) \overline{M(\xi)}}{z - \bar{\xi}}$$

is a nonnegative kernel.

- (iv) (a) *The function $M(z)$ has the expansion (2.18) at infinity.*
 (b) *The function*

$$Q(z) := \frac{i M(z)}{i + \operatorname{Im} b M(z)}$$

has analytic continuation onto the outside of some bounded interval of the real axis as the Herglotz–Nevanlinna function.

- (v) (a) *The function $M(z)$ has the expansion (2.18) at infinity.*
 (b) *The function*

$$\mathfrak{M}(z) := -\frac{1}{M(z)} + b$$

has analytic continuation onto the outside of some bounded interval of the real axis as the Herglotz–Nevanlinna function.

Proof. Let $M(z)$ be the Weyl function of some bounded prime dissipative operator A with a rank one imaginary part. From (2.11) we get

$$\begin{aligned} M(z) &= ((A - zI)^{-1}e, e) = -\sum_{k=0}^{\infty} \frac{(A^k e, e)}{z^{k+1}} \\ &= \frac{1}{z} - \frac{b}{z^2} + O\left(\frac{1}{z^3}\right), \quad |z| > \|A\|, \end{aligned}$$

where $e \in \operatorname{ran}(A - A^*)$, $\|e\| = 1$, $b = (Ae, e)$, and $\operatorname{Im} b > 0$. So, (2.18) holds. Since $A_I e = \operatorname{Im} b e$, from (2.5) it follows that the function $W(z) = 1 - 2i \operatorname{Im} b M(z)$ coincides with the characteristic function of A . Hence

$$\mathcal{K}(z, \xi) = \frac{M(z) - \overline{M(\xi)} + \operatorname{Im} b M(z) \overline{M(\xi)}}{z - \bar{\xi}} = \frac{1 - W(z) \overline{W(\xi)}}{i(z - \bar{\xi})}$$

is a nonnegative kernel. By (2.14) the function

$$Q(z) = \frac{i M(z)}{i + \operatorname{Im} b M(z)}, \quad z \in \rho(A) \cap \rho(A_R)$$

coincides with the Weyl function of A_R and therefore, has analytic continuation onto the outside of some bounded interval of the real axis as the Herglotz–Nevanlinna function. By (2.17), the function

$$\mathfrak{M}(z) = -\frac{1}{M(z)} + b$$

also has analytic continuation onto the outside of some bounded interval of the real axis as the Herglotz–Nevanlinna function. Thus, the statement (i) implies the statements (ii), (iii), (iv), and (v).

Now suppose that the statement (ii) holds true, i.e. there exist a Hilbert space \mathcal{H} and a bounded prime dissipative operator A in \mathcal{H} with a rank one imaginary part such that

$$1 - i((A - zI)^{-1}g, g) = 1 - 2i\operatorname{Im} bM(z),$$

where $g \in \operatorname{ran} A_I$ and $A_I = (\cdot, g)g$. It follows that $\|g\|^2 = 2\operatorname{Im} b$ and

$$M(z) = ((A - zI)^{-1}e, e), \quad \text{where } e = \frac{g}{\sqrt{2\operatorname{Im} b}}.$$

By definition 2.8 the function $M(z)$ is the Weyl function of A . So, (ii) \Rightarrow (i). If (iii) holds, then for $W(z) = 1 - 2i\operatorname{Im} bM(z)$ from (2.19) we obtain that

$$\mathcal{K}(z, \xi) = \frac{1 - W(z)\overline{W(\xi)}}{i(z - \bar{\xi})}.$$

Since $\mathcal{K}(z, \xi)$ is a nonnegative kernel, we get that $|W(z)| \leq 1$, $\operatorname{Im} z < 0$, and $|W(z)| = 1$ for $z \in \mathbb{C} \setminus \mathcal{D}$, $\operatorname{Im} z = 0$. In addition, from (2.18) we get that $\lim_{z \rightarrow \infty} W(z) = 1$. By Theorem 2.5 the function $W(z)$ is the characteristic function of some bounded prime dissipative operator with rank one imaginary part. Thus (iii) \Rightarrow (ii).

Let us show that (iv) \Rightarrow (ii). Define the function

$$W(z) = \frac{1 - i\operatorname{Im} b Q(z)}{1 + i\operatorname{Im} b Q(z)}.$$

Then $W(z) = 1 - 2i\operatorname{Im} bM(z)$. Since $Q(z)$ is a Herglotz–Nevanlinna function, we have $|W(z)| \leq 1$, $\operatorname{Im} z < 0$, $|W(z)| = 1$ for $z \in \mathbb{C} \setminus \mathcal{D}$, $\operatorname{Im} z = 0$, and $\lim_{z \rightarrow \infty} W(z) = 1$. By Theorem 2.5 the function $W(z)$ is the characteristic function of some bounded prime dissipative operator with a rank one imaginary part.

Suppose that (v) holds true. Because

$$\mathfrak{M}(z) = -\frac{1}{M(z)} + b$$

is a Herglotz–Nevanlinna function, the function

$$\frac{\mathfrak{M}(z) - \overline{\mathfrak{M}(\xi)}}{z - \bar{\xi}}$$

is a nonnegative kernel. From (2.19) it follows that

$$\mathcal{K}(z, \xi) = M(z)\overline{M(\xi)} \frac{\mathfrak{M}(z) - \overline{\mathfrak{M}(\xi)}}{z - \bar{\xi}}.$$

Therefore, $\mathcal{K}(z, \xi)$ is a nonnegative kernel and (v) \Rightarrow (ii). \square

3. JACOBI MATRICES AND THEIR WEYL FUNCTIONS

3.1. Complex Jacobi matrices and corresponding Jacobi operators. Let $\{b_k\}$, $\{a_k\}$, $a_k \neq 0$, $k \geq 1$ be complex numbers. A tri-diagonal matrix of the form

$$(3.1) \quad J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & a_{n-1} & b_n \end{pmatrix}$$

is called finite complex Jacobi matrix and a matrix

$$(3.2) \quad J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

is called semi-infinite complex Jacobi matrix. The case of real entries $\{b_k\}$ and positive entries $\{a_k\}$ corresponds to the classical symmetric Jacobi matrix [1], [29], [33]. We will call such matrix a self-adjoint Jacobi matrix. A finite $n \times n$ Jacobi matrix determines a linear operator (the Jacobi operator) in the Hilbert space \mathbb{C}^n . Let \mathcal{C}_0 be the linear manifold of vectors in $l_2(\mathbb{N})$ with finite support. A semi-infinite Jacobi matrix determines two linear operators in $l_2(\mathbb{N})$ given by the formal matrix product $J \cdot x$. The first operator is defined in \mathcal{C}_0 . This operator is densely defined and is closable. Let $[J]_{min}$ be its closure. The second operator $[J]_{max}$ has the domain

$$\text{dom}([J]_{max}) = \{x \in l_2(\mathbb{N}) : J \cdot x \in l_2(\mathbb{N})\}.$$

A semi-infinite Jacobi matrix is called proper if $[J]_{min} = [J]_{max}$ (cf. [6]). It is well known (cf. [2], [8]) that $\text{dom}[J]_{min} = l_2(\mathbb{N})$ and $[J]_{min}$ is bounded in $l_2(\mathbb{N})$ if and only if the entries $\{a_k\}$ and $\{b_k\}$ are uniformly bounded, i.e. condition (1.3) is fulfilled. The tri-diagonal matrix is compact if and only if

$$\lim_{k \rightarrow \infty} b_k = \lim_{k \rightarrow \infty} a_k = 0.$$

A Jacobi matrix is called bounded if (1.3) holds.

Because

$$(3.3) \quad \begin{aligned} (J^k \delta_1)_{k+1} &= a_k a_{k-1} \dots a_1, \\ (J^k \delta_1)_m &= 0, \quad m \geq k+2, \end{aligned}$$

and $a_k \neq 0$, the vectors $\delta_1, J\delta_1, \dots, J^k\delta_1, \dots$ are linearly independent and moreover the vector δ_1 is cyclic for the operator $[J]_{min}$ in $l_2(\mathbb{N})$.

The system of second order difference equations

$$(3.4) \quad \begin{aligned} a_k P_{k+1}(z) + b_k P_k(z) + a_{k-1} P_{k-1}(z) &= z P_k(z), \quad k \geq 1, \quad a_0 := 1, \\ P_0(z) &= 0, \quad P_1(z) = 1 \end{aligned}$$

determines polynomials $P_k(z)$ of degree $k - 1$ (in the case of $n \times n$ Jacobi matrix we define $a_n := 1$). Note that

$$(3.5) \quad P_{k+1}(z) = \frac{1}{a_1 \cdots a_k} z^k + \text{lower degree in } z, \quad k = 1, \dots, n.$$

3.2. Finite complex Jacobi matrices. The next proposition probably is well-known. We give a proof for completeness.

Proposition 3.1. *Let J be an $n \times n$ complex Jacobi matrix with $a_k \neq 0$ for all k and let polynomial $\{P_k(z)\}_0^{n+1}$ be defined by (3.4). Then z_0 is an eigenvalue of J if and only if $P_{n+1}(z_0) = 0$. Moreover, if z_0 is a root of the polynomial $P_{n+1}(z)$ of the multiplicity l , then vectors in \mathbb{C}^n defined as follows*

$$(3.6) \quad \begin{aligned} e_0 &= \begin{pmatrix} 1 \\ P_2(z_0) \\ \vdots \\ P_n(z_0) \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ P'_2(z_0) \\ \vdots \\ P'_n(z_0) \end{pmatrix}, \quad e_2 = \frac{1}{2!} \begin{pmatrix} 0 \\ 0 \\ P''_3(z_0) \\ \vdots \\ P''_n(z_0) \end{pmatrix}, \quad \dots, \\ e_{l-1} &= \frac{1}{(l-1)!} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ P_l^{(l-1)}(z_0) \\ \vdots \\ P_n^{(l-1)}(z_0) \end{pmatrix} \end{aligned}$$

satisfy the relations

$$(J - z_0 I)e_0 = 0, \quad (J - z_0 I)e_{k+1} = e_k, \quad k = 0, 1, \dots, l-2.$$

In addition, the relation

$$(3.7) \quad P_{k+1}(z) = a_1 a_2 \cdots a_k \det(zI - J_{[1,k]}), \quad k = 1, \dots, n$$

holds, where $J_{[1,k]}$ is the $k \times k$ upper left corner of J .

[illegible]

The same is true for $\det(J_{[1,k]} - zI)$ and for the polynomial $P_{k+1}(z)$, $k = 0, 1, \dots, n$. Taking into account (3.5), we get that (3.7) holds. \square

$$a_{n-1}P_{n-1}(z_0) + b_nP_n(z_0) + a_{n+1}P_{n+1}(z_0) = z_0P_n(z_0),$$

3.3. Semi-infinite Jacobi matrices. In the following we will consider bounded Jacobi semi-infinite matrices of the form (3.2). For the corresponding Jacobi operator

$$(3.9) \quad \sum_{k=1}^{\infty} |P_k(z)|^2 < \infty$$
$$\rho(J) = \left\{ z \in \mathbb{C} : \sup_{n \geq 1} \frac{\sum_{k=1}^n |P_k(z)|^2}{|a_n|^2 [|P_n(z)|^2 + |P_{n+1}(z)|^2]} < \infty \right\}.$$

The function

$$(3.10) \quad m_J(z) = ((J - zI)^{-1}\delta_1, \delta_1)$$

is called the *Weyl function* of J . Note that the definition of the Weyl function in the form (3.10) for a non-self-adjoint Jacobi matrix is given in [6]. The Weyl function has the following Taylor expansion at infinity

$$m_J(z) = - \sum_{l=0}^{\infty} \frac{(J^l \delta_1, \delta_1)}{z^{l+1}}$$

and the continued fraction (J -fractions) expansion [23], [34]

$$(3.11) \quad m_J(z) = \frac{-1}{z - b_1} + \frac{-a_1^2}{z - b_2} + \frac{-a_2^2}{z - b_3} + \dots + \frac{-a_{n-1}^2}{z - b_n} + \dots.$$

It follows that

$$m_J(z) \sim -\frac{1}{z} - \frac{b_1}{z^2} - \frac{b_1^2 + a_1^2}{z^3} + O\left(\frac{1}{z^4}\right), \quad z \rightarrow \infty.$$

3.4. The m -function approach in inverse spectral problems for self-adjoint Jacobi matrices. It is well-known [19] that a finite real $n \times n$ Jacobi matrix J has n distinct real eigenvalues. A bounded real semi-infinite Jacobi matrix determines a bounded self-adjoint Jacobi operator in $l_2(\mathbb{N})$ with the simple spectrum. If $E(t)$ is the orthogonal resolution of identity for the operator J , then $d\rho(t) = d(E(t)\delta_1, \delta_1)$ is a probability measure supported at n points in the case of an $n \times n$ Jacobi matrix and on a finite interval of the real axis in the semi-infinite case. The probability measure $d\rho$ is called the *spectral measure* of J . The corresponding Weyl function

$$m_J(z) = \int \frac{d\rho(t)}{t - z}.$$

belongs to Herglotz-Nevanlinna class.

The inverse spectral problems for finite and semi-infinite self-adjoint Jacobi matrices were studied in [15], [19], [20], [21], [22], [32]. The following theorem is established by M. Stone [32] for possibly unbounded self-adjoint operator.

Theorem 3.3. [1], [32]. *Let A be a self-adjoint operator with simple spectrum in a separable Hilbert space \mathcal{H} . Then there exists an orthonormal basis in \mathcal{H} in which the matrix of A is a Jacobi matrix with the conditions*

$$(3.12) \quad b_k \text{ are real numbers, } a_k > 0 \text{ for all } k$$

The construction of the corresponding self-adjoint Jacobi matrix can be provided by orthogonal normalization of $1, x, x^2, \dots$ with respect to the measure $d\rho$ or by means of m -functions (see [15]). We briefly describe the later approach for finite matrix and will keep notations of [15]. Denote by $J_{[k,n]}$, $k = 2, \dots, n$ the Jacobi matrix obtained from J by deleting $k-1$ top rows and $k-1$ left columns of J , $J_{[1,n]} = J$. Let δ_k be $k \times 1$ column $\delta_k = (0 \ \dots \ 0 \ 1 \ 0 \ \dots \ 0)^T$. Define the functions

$$(3.13) \quad m_+(z, k-1) = ((J_{[k,n]} - zI)^{-1} \delta_k, \delta_k), \quad k = 1, \dots, n-1.$$

Thus $m_+(z, 0) = m_J(z)$. As in [15] we will use the notation $m_+(z)$ for the function $m_J(z)$.

Since the functions $m_+(z, k-1)$ has the expansion in a neighborhood of infinity

$$m_+(z, k-1) = - \sum_{l=0}^{\infty} \frac{(J_{[k,n]}^l \delta_k, \delta_k)}{z^{l+1}},$$

from (3.1) it follows that

$$(3.14) \quad \begin{aligned} m_+(z, k-1) &\sim -\frac{1}{z} - \frac{b_k}{z^2} - \frac{b_k^2 + a_k^2}{z^3} + O\left(\frac{1}{z^4}\right), \quad k = 1, \dots, n-1, \\ m_+(z, n-1) &\sim -\frac{1}{z} - \frac{b_n}{z^2} - \frac{b_n^2}{z^3} + O\left(\frac{1}{z^4}\right), \end{aligned}$$

and (2.16) yields the following relations

$$(3.15) \quad a_k^2 m_+(z, k) + \frac{1}{m_+(z, k-1)} = b_k - z, \quad k = 1, \dots, n.$$

Theorem 3.4. *Every n -point probability measure is the spectral measure of a unique $n \times n$ Jacobi matrix.*

Proof. We will follow the approach considered in [15]. Let $d\rho$ be a probability measure supported at n real points t_1, \dots, t_n . Define the Herglotz-Nevanlinna function

$$m(z) = \int \frac{d\rho(t)}{t-z} = \sum_{k=1}^n \frac{\alpha_k}{t_k - z},$$

where $\alpha_k > 0$ and $\sum_{k=1}^n \alpha_k = 1$. Then in the neighborhood of infinity one has

$$m(z) \sim -\frac{1}{z} - \frac{b_1}{z^2} - \frac{b_1^2 + a_1^2}{z^3} + O\left(\frac{1}{z^4}\right),$$

where

$$b_1 = \sum_{k=1}^n \alpha_k t_k, \quad b_1^2 + a_1^2 = \sum_{k=1}^n \alpha_k t_k^2.$$

One can find also b_1 and $a_1 > 0$ by

$$b_1 = - \lim_{z \rightarrow \infty} z^2 \left(m(z) + \frac{1}{z} \right),$$

$$a_1^2 = -b_1^2 - \lim_{z \rightarrow \infty} z^3 \left(m(z) + \frac{1}{z} + \frac{b_1^2}{z^2} \right).$$

Define the Herglotz-Nevanlinna function $m_+(z, 1)$ by

$$m_+(z, 1) = \frac{1}{a_1^2} \left(b_1 - z - \frac{1}{m(z)} \right).$$

This function has an expansion

$$m_+(z, 1) \sim -\frac{1}{z} - \frac{b_2}{z^2} - \frac{b_2^2 + a_2^2}{z^3} + O\left(\frac{1}{z^4}\right)$$

in the neighborhood of infinity. Find b_2 and $a_2 > 0$ by means of limits at infinity and define $m_+(z, 2)$ similarly. Continuing this process up to

$$m_+(z, n-1) = \frac{1}{a_{n-1}^2} \left(b_{n-1} - z - \frac{1}{m_+(z, n-2)} \right) = \frac{1}{b_n - z}$$

and finding b_n by

$$b_n = - \lim_{z \rightarrow \infty} z^2 \left(m_+(z, n-1) + \frac{1}{z} \right),$$

we obtain the real numbers $\{b_k\}_{k=1}^n$ and the positive numbers $\{a_k\}_{k=1}^{n-1}$. Let us construct a self-adjoint Jacobi matrix J of the form (3.1). Then the m -function

$$m_+(z, 0) = ((J - zI)^{-1} \delta_1, \delta_1)$$

coincides with $m(z)$. □

Remark 3.5. *The following J -fraction expansion of the function $m_+(z)$*

$$(3.16) \quad m_+(z) = \frac{-1}{z - b_1} + \frac{-a_1^2}{z - b_2} + \frac{-a_2^2}{z - b_3} + \dots + \frac{-a_{n-1}^2}{z - b_n}$$

holds.

For semi-infinite Jacobi matrices with conditions (3.12) and (1.3) the approach in [15] is based on the representation of $m(z)$ as a continued J -fraction of the form (3.11) and the following result:

Theorem 3.6. [15]. *Suppose that $m(z) = \int (t - z)^{-1} d\rho(t)$, where $d\rho$ is a probability measure on $[-C, C]$ whose support contains more than one point. Let*

$$b_1 = \int d\rho(t), \quad a_1^2 = \int t^2 d\rho(t) - b_1^2,$$

and let

$$m_1(z) = \frac{1}{a_1^2} \left(b_1 - z - \frac{1}{m(z)} \right).$$

Then $m_1(z) = \int (t - z)^{-1} d\rho_1(t)$, where $d\rho_1$ is a probability measure also supported on $[-C, C]$.

We give one more application of the m -function approach which we will use later.

Proposition 3.7. *Let $m(z)$ be a Herglotz-Nevanlinna function which is odd in some neighborhood of infinity and $\lim_{z \rightarrow \infty} zm(z) = -1$. Then the diagonal entries of the corresponding self-adjoint Jacobi matrix are equal zero.*

Proof. Since $m(-z) = -m(z)$, the function $m(z)$ has the following Taylor's expansion at infinity

$$(3.17) \quad m(z) = - \sum_{k=0}^{\infty} \frac{\beta_k}{z^{2k+1}}, \quad \beta_0 = 1.$$

Let J be a bounded self-adjoint Jacobi matrix whose Weyl's function coincides with $m(z)$. Then due to (3.17) we obtain $b_1 = 0$. By (3.15) we have

$$m_+(z, 1) = \frac{1}{a_1^2} \left(-z - \frac{1}{m(z)} \right).$$

It follows that $m_+(-z, 1) = -m_+(z, 1)$. Now (3.14) yields $b_2 = 0$. Hence by induction and again using (3.15) and (3.14), we obtain that $b_k = 0$ for all k . \square

3.5. Mixed given data and uniqueness for finite self-adjoint Jacobi matrixes. Let J be an $n \times n$ self-adjoint Jacobi matrix. Following [15] we will consider b 's and a 's as a single sequence $\{c_k\}_{k=1}^{2n-1}$, where $c_{2k-1} = b_k$ and $c_{2k} = a_k$. The next theorem is established by H. Hochstadt in [21].

Theorem 3.8. [21]. *Let J be an $n \times n$ self-adjoint Jacobi matrix. Suppose that c_{n+1}, \dots, c_{2n-1} are known as well as the eigenvalues z_1, \dots, z_n of J . Then c_1, \dots, c_n are uniquely determined.*

F. Gesztesy and B. Simon in [15] proved the following generalization of the Hochstadt theorem.

Theorem 3.9. [15]. *Let J be an $n \times n$ self-adjoint Jacobi matrix. Suppose that c_{j+1}, \dots, c_{2n-1} are known as well as j of the eigenvalues. Then c_1, \dots, c_j are uniquely determined.*

Note that it is not necessary to know which of the j eigenvalues one has, and there may be no matrix consistent with the data as in Theorems 3.8 and 3.9 (see [13]).

Since later on we will consider a non-self-adjoint analog of the Hochstadt and Gesztesy-Simon theorems, let us mention the key moment of the proof of Theorem 3.9 in [15]. Apart from $m_+(z, k)$ -functions the following m_- - functions are used in [15]:

$$(3.18) \quad m_-(z, k) = ((J_{[1, k-1]} - zI)^{-1} \delta_{k-1}, \delta_{k-1}), \quad k = 2, 3, \dots, n.$$

It is established in [15] that for any eigenvalue λ_j of a self-adjoint $n \times n$ Jacobi matrix J and for any $k = 1, \dots, n$ the equality

$$(3.19) \quad m_-(\lambda_j, k+1) = [a_k^2 m_+(\lambda_j, n)]^{-1}$$

holds, where the equality in (3.19) includes the case that both sides equal infinity.

4. DISSIPATIVE JACOBI MATRICES WITH A RANK ONE IMAGINARY PART

4.1. Finite and semi-infinite dissipative Jacobi matrices. In the following we will consider the Jacobi matrix of the form (3.1) and (3.2) but now we suppose that condition (1.4) is fulfilled.

Proposition 4.1. *Let J be a non-self-adjoint $n \times n$ or semi-infinite Jacobi matrix of the form (3.1) with conditions (1.4) and (1.3). Then J defines a prime dissipative operator with a rank one imaginary part in \mathbb{C}^n ($l_2(\mathbb{N})$) and its Livsic characteristic function takes the form*

$$(4.1) \quad W(z) = 1 - 2i \operatorname{Im} b_1 ((J - zI)^{-1} \delta_1, \delta_1).$$

Proof. Let J^* be the adjoint matrix to J and let

$$J_R = \frac{1}{2}(J + J^*), \quad J_I = \frac{1}{2i}(J - J^*)$$

be Hermitian components of J . One has for $n \times n$ case

(4.2)

$$J_R = \begin{pmatrix} \operatorname{Re} b_1 & a_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & a_{n-1} & b_n \end{pmatrix}, \quad J_I = \begin{pmatrix} \operatorname{Im} b_1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

and for semi-infinite case

(4.3)

$$J_R = \begin{pmatrix} \operatorname{Re} b_1 & a_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad J_I = \begin{pmatrix} \operatorname{Im} b_1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

and for every $x \in \mathbb{C}^n$ ($l_2(\mathbb{N})$)

$$2J_I x = (x, g) g,$$

where

$$(4.4) \quad g = \sqrt{2 \operatorname{Im} b_1} \delta_1.$$

By (2.5) one obtains (4.1). From (3.3) it follows that J is prime. \square

Remark 4.2. *One can easily show that a non-self-adjoint finite or semi-infinite bounded tri-diagonal matrix \mathcal{A} with diagonal entries $\{d_k\}$, sub-diagonal entries $\{c_k\}$, and super-diagonal entries $\{\overline{c_k}\}$ has a rank one imaginary part if and only if there exists a diagonal nonzero entry d_m such that*

$$d_m d_{m+1} = |c_m|^2,$$

and remaining entries equal zero. It follows that the Jacobi matrix J with the conditions $\operatorname{Im} b_1 = \dots = \operatorname{Im} b_{m-1} = 0$, $\operatorname{Im} b_m > 0$, $\operatorname{Im} b_{m+1} = \dots = 0$, $a_k > 0$ for all k is also dissipative with a rank one imaginary part. But in this case such a matrix might not be prime. For example the 3×3 matrix of the form

$$J = \begin{pmatrix} 1 & a_1 & 0 \\ a_1 & -1 + it & a_2 \\ 0 & a_2 & 1 \end{pmatrix}$$

is dissipative for $t > 0$ and

$$\operatorname{ran} J_I = \{\lambda \delta_2, \lambda \in \mathbb{C}\}.$$

But $\operatorname{span} \{\delta_2, J\delta_2, J^2\delta_2\} \neq \mathbb{C}^3$. So, this matrix is not prime.

The next proposition easily follows from (2.6), (2.16), and (3.10).

Proposition 4.3. *Let J be a bounded non-self-adjoint finite or semi-infinite Jacobi matrix of the form (3.1) with condition (1.4). Then for the Weyl functions $m_J(z)$ and $m_{J_R}(z)$ of J and J_R and for the characteristic function $W(z)$ there are the following relations:*

$$\begin{aligned} m_J(z) &= \frac{i}{2 \operatorname{Im} b_1} (W(z) - 1), \\ m_{J_R}(z) &= \frac{i}{\operatorname{Im} b_1} \frac{W(z) - 1}{W(z) + 1}, \\ m_J(z) &= \frac{i m_{J_R}(z)}{i - \operatorname{Im} b_1 m_{J_R}(z)}, \\ m_{J_R}(z) &= \frac{i m_J(z)}{i + \operatorname{Im} b_1 m_J(z)}. \end{aligned}$$

Moreover, the function

$$\mathcal{K}_J(z, \xi) := \frac{m_J(z) - \overline{m_J(\xi)} + 2i \operatorname{Im} b_1 m_J(z) \overline{m_J(\xi)}}{z - \bar{\xi}}$$

is a nonnegative kernel, the function

$$-\frac{1}{m_J(z)} + b_1$$

is a Herglotz-Nevanlinna, and at infinity

$$\frac{1}{a_1^2} \left(-\frac{1}{m_J(z)} + b_1 - z \right) \sim -\frac{1}{z} - \frac{b_2}{z^2} - \frac{b_2^2 + a_2^2}{z^3} + O\left(\frac{1}{z^4}\right).$$

The points of a non-real spectrum of J are solutions of the equation

$$m_{J_R}(z) = \frac{i}{\operatorname{Im} b_1}, \quad \operatorname{Im} z > 0.$$

From Proposition 4.1, Theorem 2.7 and formulas (2.9), (2.10), (2.12), (2.13), and (3.10) we get the following statement.

Proposition 4.4. *Let J be a non-self-adjoint $n \times n$ Jacobi matrix of the form (3.1) with conditions (1.4) and let z_1, z_2, \dots, z_n be the eigenvalues*

of J counting algebraic multiplicity. Then the following formulas

$$\begin{aligned}
 W(z) &= \prod_{k=1}^n \frac{z - \overline{z_k}}{z - z_k}, \\
 m_J(z) &= \frac{i}{2 \operatorname{Im} b_1} \frac{\prod_{k=1}^n (z - \overline{z_k}) - \prod_{k=1}^n (z - z_k)}{\prod_{k=1}^n (z - z_k)}, \\
 m_{J_R}(z) &= \frac{i}{\operatorname{Im} b_1} \frac{\prod_{k=1}^n (z - \overline{z_k}) - \prod_{k=1}^n (z - z_k)}{\prod_{k=1}^n (z - \overline{z_k}) + \prod_{k=1}^n (z - z_k)}
 \end{aligned}
 \tag{4.5}$$

hold.

Proposition 4.5. *Let J be an $n \times n$ Jacobi matrix, $n \geq 3$, with the conditions (1.4). Then the matrices J and $J_{[1, n-2]}$ have no common eigenvalues and the relations*

$$-\frac{a_1 a_2 \cdots a_k}{(k-1)!} \operatorname{Im} P_{k+1}^{(k-1)}(0) = \operatorname{Im} b_1
 \tag{4.6}$$

hold for all $k = 1, \dots, n$, where the polynomials $\{P_k(z)\}_{k=1}^{n+1}$ are defined by (3.4).

Proof. Because all $J_{[1, k]}$ are dissipative and prime, their eigenvalues belong to \mathbb{C}_+ . Suppose that z_0 is an eigenvalue of J and $J_{[1, n-2]}$. Then by Proposition 3.1 $P_{n-1}(z_0) = P_{n+1}(z_0) = 0$. From (3.4) we have

$$a_{n-1} P_{n-1}(z_0) + (b_n - z_0) P_n(z_0) = 0.$$

Since $\operatorname{Im} z_0 > 0$ and b_n is real, we get that $P_n(z_0) = 0$, i.e. z_0 is a common eigenvalue for the matrices $J_{[1, n-2]}$ and $J_{[1, n-1]}$. By Proposition 3.2 this is impossible.

Let $\{z_j\}_{j=1}^k$ be the roots of the polynomial $P_{k+1}(z)$. Then $\operatorname{Im} z_j > 0$. From (3.7) it follows that

$$\sum_{j=1}^n z_j = \operatorname{Sp} J_{[1, k]}.$$

But $\operatorname{Im} \operatorname{Sp} J_{[1, k]} = \operatorname{Im} b_1$. On the other hand

$$\sum_{j=1}^n z_j = -\frac{a_1 a_2 \cdots a_k}{(k-1)!} P_{k+1}^{(k-1)}(0).$$

Therefore, (4.6) holds. \square

If J is a semi-infinite bounded Jacobi matrix of the form (3.2) and with conditions (1.4), then by Proposition 4.1 it determines a prime dissipative operator with rank one imaginary part. Therefore its non-real spectrum is a denumerable set of eigenvalues of finite algebraic multiplicities and from (3.9)

$$\sigma_p(J) = \left\{ z \in \mathbb{C} : \sum_{k=1}^{\infty} |P_k(z)|^2 < \infty \right\},$$

where $\{P_k(z)\}$ are corresponding polynomials defined by (3.4). By Theorem 2.3 the closure of the linear span of all corresponding root subspaces coincides with $l_2(\mathbb{N})$ if and only if

$$\sum_{z \in \sigma_p(J)} \operatorname{Im} z = \operatorname{Im} b_1$$

Example 4.6. Let J_l be the semi-infinite Jacobi matrix of the form

$$(4.7) \quad J_l = \begin{pmatrix} il & 1/2 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1/2 & 0 & 1/2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1/2 & 0 & 1/2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where $l > 0$. The real part is the Jacobi matrix

$$(4.8) \quad H_0 = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1/2 & 0 & 1/2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1/2 & 0 & 1/2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

It is known [8] that the Weyl function of H_0 is the function

$$m_{H_0}(z) = 2(\sqrt{z^2 - 1} - z), \quad z \in \mathbb{C} \setminus [-1, 1].$$

The spectrum of H_0 is continuous and coincides with the interval $[-1, 1]$. Therefore, $\|H_0\| = 1$. The Weyl function of J_l is the function

$$m_{J_l}(z) = \frac{2i(\sqrt{z^2 - 1} - z)}{i - 2l(\sqrt{z^2 - 1} - z)}.$$

The non-real spectrum of J_l we find solving the equation

$$\sqrt{z^2 - 1} - z = \frac{i}{2l}.$$

This equation has a unique solution from the open upper half-plane

$$z = i \frac{4l^2 - 1}{4}$$

for $l > 1/2$ and has no solutions for $l \in (0, 1/2]$. Thus if $l \in (0, 1/2]$ the spectrum of J coincides with $[-1, 1]$ and if $l > 1/2$ the spectrum of J_l is $\{i(4l^2 - 1)/4\} \cup [-1, 1]$.

Note that

$$\lim_{x \uparrow -1} \frac{1}{m_{H_0}(z)} = -\lim_{x \downarrow 1} \frac{1}{m_{H_0}(z)} = \frac{1}{2}.$$

Results from [25], [4], [3] yield that the Jacobi matrices of the form

$$H_x = \begin{pmatrix} x & 1/2 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1/2 & 0 & 1/2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1/2 & 0 & 1/2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

determine contractive Jacobi operators in $l_2(\mathbb{N})$ if and only if $|x| \leq 1/2$.

Example 4.7. Let \hat{J}_l be the semi-infinite Jacobi matrix of the form

$$(4.9) \quad \hat{J}_l = \begin{pmatrix} il & 1/\sqrt{2} & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1/\sqrt{2} & 0 & 1/2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1/2 & 0 & 1/2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where $l > 0$. The real part of \hat{J}_l is the Jacobi matrix

$$(4.10) \quad \hat{H}_0 = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1/\sqrt{2} & 0 & 1/2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1/2 & 0 & 1/2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

It is known [8] that the Weyl function of \hat{H}_0 is the function

$$m_{\hat{H}_0}(z) = -\frac{1}{\sqrt{z^2 - 1}}, \quad z \in \mathbb{C} \setminus [-1, 1].$$

The spectrum of \hat{H}_0 is continuous and coincides with the interval $[-1, 1]$. Therefore $\|\hat{H}_0\| = 1$. The Weyl function of \hat{J}_l is the function

$$m_{\hat{J}_l}(z) = -\frac{i}{i\sqrt{z^2 - 1} + l}.$$

The non-real spectrum of \hat{J}_l we find solving the equation

$$\sqrt{z^2 - 1} = il.$$

This equation has a unique solution from the open upper half-plane

$$z = i\sqrt{l^2 - 1}$$

for $l > 1$ and has no solutions for $l \in (0, 1]$. Thus if $l \in (0, 1]$ the spectrum of \widehat{J}_l coincides with $[-1, 1]$ and if $l > 1$ the spectrum of J_l is $\{i\sqrt{l^2 - 1}\} \cup [-1, 1]$.

Note that

$$\lim_{x \uparrow -1} |m_{\widehat{H}_0}(x)| = \lim_{x \downarrow 1} |m_{\widehat{H}_0}(x)| = +\infty.$$

From [25] it follows that the Jacobi matrix \widehat{H}_0 is a unique contractive matrix in $l_2(\mathbb{N})$ among Jacobi matrices of the form

$$\widehat{H}_x = \begin{pmatrix} x & 1/\sqrt{2} & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1/\sqrt{2} & 0 & 1/2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1/2 & 0 & 1/2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

4.2. The m -functions of a finite dissipative Jacobi matrix with a rank one imaginary part. The next proposition establishes the analogues of the relations (3.19) for a dissipative Jacobi matrix.

Theorem 4.8. *Let J be an $n \times n$ Jacobi matrix with conditions (1.4). Let $m_J(z)$ be the Weyl function of J defined by (3.10) and let the functions $m_+(z, j)$ and $m_-(z, j)$ be defined by (3.13) and (3.18). If z_0 is the eigenvalue of J of the algebraic multiplicity l then for all $j = 1, \dots, n$ the following relations hold*

$$(4.11) \quad \left(\frac{1}{m_-(z, j+1)} \right)^{(p)} \Big|_{z=z_0} = a_j^2 m_+^{(p)}(z_0, j), \quad p = 0, \dots, l-1,$$

Proof. Represent the matrix J in the block-matrix form

$$J = \begin{pmatrix} J_{[1,j]} & A_{12} \\ A_{21} & J_{[j+1,n]} \end{pmatrix},$$

where A_{12} is $j \times (n-j)$ matrix of the form

$$A_{12} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ a_j & 0 & \cdots & 0 \end{pmatrix}$$

and $A_{21} = A_{12}^*$ (is $(n-j) \times j$ matrix). Let $\mathcal{H}_1 = \mathbb{C}^j$. Then from (2.15) it follows that

$$P_{\mathcal{H}_1}(J - z)^{-1} \upharpoonright \mathcal{H}_1 = (J_{[1,j]} - zI - A_{12}(J_{[j+1,n]} - zI)^{-1}A_{21})^{-1}$$

for $z \in \rho(J) \cap \rho(J_{[j+1,n]})$. Clearly

$$A_{12}(J_{[j+1,n]} - zI)^{-1}A_{21} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & a_j^2 m_+(z, j) \end{pmatrix}.$$

Now from (2.8) we get

(4.12)

$$((J - zI)^{-1}\delta_j, \delta_j) = \frac{m_-(z, j+1)}{1 - a_j^2 m_+(z, j) m_-(z, j+1)}, \quad j = 1, 2, \dots, n.$$

Let $j \geq 2$. By Cramer's rule we have

$$((J - zI)^{-1}\delta_j, \delta_j) = \frac{\det(J_{[1,j-1]} - zI) \det(J_{[j+1,n]} - zI)}{\det(J - zI)}$$

and

$$m_-(z, j+1) = \frac{\det(J_{[1,j-1]} - zI)}{\det(J_{[1,j]} - zI)}$$

for $j = 2, \dots, n$. From (4.12) it follows that

$$\frac{\det(J_{[j+1,n]} - zI)}{\det(J - zI)} = \frac{1}{\det(J_{[1,j]} - zI) - a_j^2 m_+(z, j) \det(J_{[1,j-1]} - zI)}$$

Let z_0 be an eigenvalue of J of the algebraic multiplicity l . Because J is dissipative, we have $\text{Im } z_0 > 0$. Note that $J_{[j+1,n]}$ is self-adjoint Jacobi matrix, the Herglotz-Nevanlinna function $m_+(z, j)$ is holomorphic at z_0 and $\text{Im } m_+(z_0, j) > 0$. It follows that the function

$$\frac{\det(J_{[j+1,n]} - zI)}{\det(J - zI)}$$

has the pole at z_0 of order l . Therefore

$$\lim_{z \rightarrow z_0} (z - z_0)^p \frac{\det(J_{[j+1,n]} - zI)}{\det(J - zI)} = \infty, \quad p = 0, 1, \dots, l-1$$

and

$$\lim_{z \rightarrow z_0} (z - z_0)^l \frac{\det(J_{[j+1,n]} - zI)}{\det(J - zI)} \quad \text{is finite.}$$

This yields

$$(4.13) \quad \lim_{z \rightarrow z_0} \frac{(z - z_0)^p}{\det(J_{[1,j]} - zI) - a_j^2 m_+(z, j) \det(J_{[1,j-1]} - zI)} = \infty$$

for $p = 0, 1, \dots, l-1$. In particular, for $p = 0$ we obtain

$$(4.14) \quad \det(J_{[1,j]} - z_0 I) - a_j^2 m_+(z_0, j) \det(J_{[1,j-1]} - z_0 I) = 0.$$

By Proposition 3.2 the matrices $J_{[1,j-1]}$ and $J_{[1,j]}$ have no common eigenvalues and since $m_+(z_0, j) \neq 0$, (4.14) implies that

$$\det(J_{[1,j]} - z_0 I) \neq 0, \det(J_{[1,j-1]} - z_0 I) \neq 0$$

and

$$(4.15) \quad \frac{1}{m_-(z_0, j+1)} = \frac{\det(J_{[1,j]} - z_0 I)}{\det(J_{[1,j-1]} - z_0 I)} = a_j^2 m_+(z_0, j).$$

In some neighborhood of z_0 we can rewrite (4.13) as

$$\lim_{z \rightarrow z_0} \frac{1}{\det(J_{[1,j-1]} - z I)} \frac{(z - z_0)^p}{(m_-(z, j+1))^{-1} - a_j^2 m_+(z, j)} = \infty$$

for $p = 0, 1, \dots, l-1$. It follows that the number z_0 is a zero of the function

$$\frac{1}{m_-(z, j+1)} - a_j^2 m_+(z, j)$$

of order l . Thus, we get (4.11) for $j \geq 2$.

Let $j = 1$. Since $m_-(z, 2) = (b_1 - z)^{-1}$, relation (4.12) takes the form

$$m_J(z) = \frac{1}{b_1 - z - a_1^2 m_+(z, 1)}.$$

If z_0 is the eigenvalue of J of the algebraic multiplicity l , then due to (4.5) the Weyl function $m_J(z)$ has a pole of order l at z_0 . In the same manner as the above we get that z_0 is a zero of order l for the function

$$b_1 - z - a_1^2 m_+(z, 1) = \frac{1}{m_-(z, 2)} - a_1^2 m_+(z, 1).$$

Therefore, (4.11) holds and for $j = 1$. □

Since the integer j is an arbitrary, we obtain as a by-product the following corollary.

Corollary 4.9. *Let J be an $n \times n$ Jacobi matrix with the conditions (1.4). Then the matrices $J_{[1,j]}$ and J have no common eigenvalues for every $j = 1, 2, \dots, n-1$.*

5. INVERSE SPECTRAL PROBLEMS FOR FINITE JACOBI MATRICES WITH A RANK ONE IMAGINARY PART

5.1. Reconstruction of a finite dissipative Jacobi matrix from its eigenvalues. Let J be a non-self-adjoint $n \times n$ Jacobi matrix of the form (3.1) satisfying conditions (1.4). Then by Proposition 4.1 the corresponding operator in \mathbb{C}^n is a prime dissipative operator with a rank one imaginary part. Therefore the matrix J has only non-real

eigenvalues with positive imaginary parts. The next theorem establishes that an arbitrary n non-real numbers counting multiplicity from the open upper half-plane determine uniquely some dissipative $n \times n$ Jacobi matrix with a rank one imaginary part.

Theorem 5.1. *Suppose that z_1, \dots, z_n are not necessarily distinct complex numbers with positive imaginary parts. Then there exists a unique $n \times n$ Jacobi matrix with entries satisfying conditions (1.4) whose eigenvalues counting algebraic multiplicity coincide with $\{z_k\}_{k=1}^n$.*

Proof. Let

$$(5.1) \quad W(z) = \prod_{k=1}^n \frac{z - \bar{z}_k}{z - z_k}.$$

Then

$$\lim_{z \rightarrow \infty} z(W(z) - 1) = 2i \sum_{k=1}^n \operatorname{Im} z_k.$$

Let

$$c = \sum_{k=1}^n \operatorname{Im} z_k.$$

Define

$$(5.2) \quad m_+(z, 0) = \frac{i}{c} \frac{W(z) - 1}{W(z) + 1}.$$

The Herglotz-Nevanlinna function $m_+(z, 0)$ has the expansion at infinity

$$m_+(z, 0) \sim -\frac{1}{z} - \frac{b}{z^2} - \frac{b^2 + a_1^2}{z^3} + O\left(\frac{1}{z^4}\right),$$

and determines a probability measure supported at some n points. By Theorem 3.4 there exists a unique self-adjoint $n \times n$ Jacobi matrix

$$H = \begin{pmatrix} b & a_1 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & 0 & a_{n-1} & b_n \end{pmatrix}$$

satisfying conditions (3.12) such that

$$m_+(z, 0) = ((H - zI)^{-1}\delta_1, \delta_1).$$

Let

$$J = \begin{pmatrix} b+ic & a_1 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & 0 & a_{n-1} & b_n \end{pmatrix} = H + i \begin{pmatrix} c & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \end{pmatrix}.$$

It follows that

$$2 J_I x = (x, g), \quad x \in \mathbb{C}^n,$$

where $g = \sqrt{2c}\delta_1 \in \mathbb{C}^n$.

By Proposition 4.1 for the characteristic function of J we obtain

$$w(z) = 1 - 2i c \left((J - zI)^{-1} \delta_1, \delta_1 \right).$$

Its fractional-linear transformation

$$v(z) = i \frac{w(z) - 1}{w(z) + 1}$$

has the form

$$v(z) = \frac{1}{2} \left((H - zI)^{-1} g, g \right) = c \left((H - zI)^{-1} \delta_1, \delta_1 \right) = c m_+(z, 0).$$

From (5.2) we get

$$v(z) = i \frac{W(z) - 1}{W(z) + 1}.$$

Therefore $w(z) = W(z)$. By Theorem 2.7 the eigenvalues of J counting algebraic multiplicity coincide with $\{z_k\}$. \square

Example 5.2. Let us construct 3×3 dissipative Jacobi matrix with eigenvalues $z_1 = i$ of multiplicity 2 and $z_2 = 2i$ (of multiplicity 1). By Theorem 2.6 the characteristic function of J is

$$W(z) = \left(\frac{z+i}{z-i} \right)^2 \frac{z+2i}{z-2i}.$$

Then

$$\lim_{z \rightarrow \infty} z(W(z) - 1) = 4i.$$

Note that $2\operatorname{Im} z_1 + \operatorname{Im} z_2 = 4$. Let

$$\begin{aligned} m_+(z, 0) &= \frac{i}{4} \frac{W(z) - 1}{W(z) + 1} = \frac{i}{4} \frac{(z+i)^2(z+2i) - (z-i)^2(z-2i)}{(z+i)^2(z+2i) + (z-i)^2(z-2i)} = \\ &= \frac{-2z^2 + 1}{2z^3 - 10z}. \end{aligned}$$

Then

$$m(z) = \frac{-1}{z - \frac{9/2}{z - \frac{1/2}{z}}}.$$

It follows that $b_1 = 4i$, $b_2 = b_3 = 0$, $a_1 = \sqrt{9/2}$, $a_2 = \sqrt{1/2}$. The Jacobi matrix takes the form

$$J = \begin{pmatrix} 4i & \frac{3}{\sqrt{2}} & 0 \\ \frac{3}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

Example 5.3. In order to construct the dissipative $n \times n$ Jacobi matrix with the eigenvalue $z_0 = x_0 + iy_0$ of algebraic multiplicity n ($y_0 > 0$) it is sufficient to construct the $n \times n$ Jacobi matrix J_n with the eigenvalue $z_0 = i$ of algebraic multiplicity n . The characteristic function of such a matrix is

$$W_n(z) = \left(\frac{z+i}{z-i} \right)^n.$$

The corresponding Weyl function of the real part is

$$m_n(z) = \frac{i}{n} \frac{(z+i)^n - (z-i)^n}{(z+i)^n + (z-i)^n}.$$

Because $\text{Sp}J = in$, we get $\text{Im } b_1 = n$. Expanding the function $m_n(z)$ via continued fraction we obtain the real part of J_n . In particular

$$\begin{aligned} m_2(z) &= \frac{-1}{z - \frac{1}{z}}, \quad m_3(z) = \frac{-1}{z - \frac{8/3}{z - \frac{1/3}{z}}}, \\ m_4(z) &= \frac{-1}{z + \frac{-5}{z + \frac{-4/5}{z + \frac{-1/5}{z}}}}, \quad m_5(z) = \frac{-1}{z + \frac{-8}{z + \frac{-7/5}{z + \frac{-16/35}{z + \frac{-1/7}{z}}}}}. \end{aligned}$$

Therefore

$$J_2 = \begin{pmatrix} 2i & 1 \\ 1 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 3i & \frac{2\sqrt{2}}{\sqrt{3}} & 0 \\ \frac{2\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & 0 \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 4i & \sqrt{5} & 0 & 0 \\ \sqrt{5} & 0 & \frac{2}{\sqrt{5}} & 0 \\ 0 & \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 0 & \frac{1}{\sqrt{5}} & 0 \end{pmatrix}, \quad J_5 = \begin{pmatrix} 5i & \frac{2\sqrt{2}}{\sqrt{5}} & 0 & 0 & 0 \\ \frac{2\sqrt{2}}{\sqrt{5}} & 0 & \frac{\sqrt{7}}{\sqrt{5}} & 0 & 0 \\ 0 & \frac{\sqrt{7}}{\sqrt{5}} & 0 & \frac{4}{\sqrt{35}} & 0 \\ 0 & 0 & \frac{4}{\sqrt{35}} & 0 & \frac{1}{\sqrt{7}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{7}} & 0 \end{pmatrix}.$$

Note that the eigenvalues of the real part of J_n are solutions of the equations

$$\left(\frac{z+i}{z-i} \right)^n = -1,$$

that are real numbers

$$\lambda_k = -\cot \left(\frac{\pi + 2\pi k}{2n} \right), \quad k = 0, 1, \dots, n-1.$$

5.2. The connection between triangular and Jacobi forms of a prime dissipative operator with a rank one imaginary part.

Let z_1, z_2, \dots, z_n be not necessarily distinct complex number with positive imaginary parts. According to Theorem 5.1 there exists a unique $n \times n$ Jacobi matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & 0 & a_{n-1} & b_n \end{pmatrix}$$

with entries satisfying conditions (1.4) whose eigenvalues coincide with $\{z_k\}_{k=1}^n$ counting algebraic multiplicity. On the other hand such a matrix by Theorem 2.7 is unitary equivalent to the triangular matrix of the form

$$(5.3) \quad \vec{J} = \begin{pmatrix} z_1 & i\beta_1\beta_2 & \cdot & \cdot & \cdot & i\beta_1\beta_n \\ 0 & z_2 & \cdot & \cdot & \cdot & i\beta_2\beta_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & z_n \end{pmatrix},$$

where

$$(5.4) \quad z_k = \alpha_k + i\frac{\beta_k^2}{2} = \operatorname{Im} z_k, \quad \beta_k > 0, \quad k = 1, \dots, n.$$

From (5.3) it follows that

$$2 \operatorname{Im} \vec{J} = \begin{pmatrix} \beta_1^2 & \beta_1\beta_2 & \cdot & \cdot & \cdot & \beta_1\beta_n \\ \beta_1\beta_2 & \beta_2^2 & \cdot & \cdot & \cdot & \beta_2\beta_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta_1\beta_n & \beta_2\beta_n & \cdot & \cdot & \cdot & \beta_n^2 \end{pmatrix}$$

and

$$(5.5) \quad 2 \operatorname{Im} \vec{J} = (\cdot, \vec{g}) \vec{g},$$

where

$$(5.6) \quad \vec{g} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_n \end{pmatrix}.$$

Let U be unitary matrix

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdot & \cdot & \cdot & u_{1n} \\ u_{21} & u_{22} & \cdot & \cdot & \cdot & u_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ u_{n1} & u_{n2} & \cdot & \cdot & \cdot & u_{nn} \end{pmatrix}$$

such that

$$(5.7) \quad \begin{cases} U J = \vec{J} U \\ U g = \vec{g} \end{cases},$$

where $g = \sqrt{2 \operatorname{Im} b_1} \delta_1$. Next we present *an algorithm which allows to find the Jacobi matrix J and the unitary matrix U satisfying (5.7)*. From (5.7) it follows

$$(5.8) \quad U J^k g = (\vec{J})^k \vec{g}, \quad k = 1, 2, \dots, n.$$

Since U is the unitary matrix, we have $\|g\|^2 = \|\vec{g}\|^2$, and taking into account (4.4) and (5.6) we get

$$\|g\|^2 = 2 \operatorname{Im} b_1 = \beta_1^2 = \beta_2^2 + \dots + \beta_n^2 = \|\vec{g}\|^2.$$

Thus

$$(5.9) \quad \operatorname{Im} b_1 = \sum_{k=1}^n \operatorname{Im} z_k.$$

From $Ug = \vec{g}$ it follows

$$(5.10) \quad u_{k1} = \frac{\sqrt{\operatorname{Im} z_k}}{\sqrt{\sum_{j=1}^n \operatorname{Im} z_j}}, \quad k = 1, 2, \dots, n.$$

Relations (5.7) and (5.8) yield

$$(\vec{J}\vec{g}, \vec{g}) = (UJg, Ug) = (Jg, g).$$

Taking into account (1.1), (4.4), (5.3) and (5.6), we get

$$(5.11) \quad Jg = \begin{pmatrix} b_1 \sqrt{2 \operatorname{Im} b_1} \\ a_1 \sqrt{2 \operatorname{Im} b_1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \quad \vec{J}\vec{g} = \begin{pmatrix} \left(z_1 + i \sum_{k=2}^n \beta_k^2 \right) \beta_1 \\ \left(z_2 + i \sum_{k=3}^n \beta_k^2 \right) \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ z_n \beta_n \end{pmatrix}.$$

Therefore

$$(Jg, g) = 2 b_1 \operatorname{Im} b_1, \\ (\vec{J}\vec{g}, \vec{g}) = \sum_{j=1}^n \left(z_j + i \sum_{k=j+1}^n \beta_k^2 \right) \beta_j^2.$$

Since $(Jg, g) = (\vec{J}\vec{g}, \vec{g})$, from (5.4) and (5.9) we get

$$b_1 = \frac{\sum_{j=1}^n \left(z_j + i \sum_{k=j+1}^n \beta_k^2 \right) \operatorname{Im} z_j}{\sum_{k=1}^n \operatorname{Im} z_k}, \quad \operatorname{Re} b_1 = \frac{\sum_{k=1}^n \operatorname{Re} z_k \operatorname{Im} z_k}{\sum_{k=1}^n \operatorname{Im} z_k}.$$

Because U is unitary and $UJg = \vec{J}\vec{g}$, we get $\|Jg\|^2 = \|\vec{J}\vec{g}\|^2$. This equality and (5.11) yield

$$|b_1|^2 (2 \operatorname{Im} b_1) + a_1^2 (2 \operatorname{Im} b_1) = \sum_{j=1}^n \left| z_j + i \sum_{k=j+1}^n \beta_k^2 \right|^2 \beta_j^2$$

and hence

$$a_1 = \left(\frac{\sum_{j=1}^n \left| z_j + i \sum_{k=j+1}^n \beta_k^2 \right|^2 \operatorname{Im} z_j}{\sum_{k=1}^n \operatorname{Im} z_k} - \frac{\left| \sum_{j=1}^n \left(z_j + i \sum_{k=j+1}^n \beta_k^2 \right) \operatorname{Im} z_j \right|^2}{\left(\sum_{k=1}^n \operatorname{Im} z_k \right)^2} \right)^{1/2}.$$

Recall that by (3.3)

$$(J^{m-1}g)_m = a_{m-1}a_{m-2} \cdots a_1 \sqrt{2 \operatorname{Im} b_1}$$

and

$$(J^{m-1}g)_{m+1} = (J^{m-1}g)_{m+2} = \cdots = (J^{m-1}g)_n = 0$$

for $m = 1, \dots, n$. Let $m \geq 2$. Suppose that $(J^{m-1}g)_k$ are already known, where $k = 1, 2, \dots, m-1$. Then from (1.1) we obtain

$$J^m g = \begin{pmatrix} b_1(J^{m-1}g)_1 + a_1(J^{m-1}g)_2 \\ a_1(J^{m-1}g)_1 + b_2(J^{m-1}g)_2 + a_2(J^{m-1}g)_3 \\ a_2(J^{m-1}g)_2 + b_3(J^{m-1}g)_3 + a_3(J^{m-1}g)_4 \\ \vdots \\ \vdots \\ \vdots \\ a_{m-1}(J^{m-1}g)_{m-1} + b_m(J^{m-1}g)_m \\ a_m(J^{m-1}g)_m \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

It follows that

$$\begin{aligned} (J^m g, J^{m-1}g)_{\mathbb{C}^n} &= \sum_{j=1}^n (J^m g)_j \overline{(J^{m-1}g)_j} = \sum_{j=1}^{m-1} (J^m g)_j \overline{(J^{m-1}g)_j} + \\ &+ (a_{m-1}(J^{m-1}g)_{m-1} + b_m(J^{m-1}g)_m) \overline{(J^{m-1}g)_m}. \end{aligned}$$

From

$$(J^m g, J^{m-1}g) = \left((\vec{J})^m \vec{g}, (\vec{J})^{m-1} \vec{g} \right)$$

and $(J^{m-1}g)_m = a_{m-1}a_{m-2} \cdots a_1 \sqrt{2 \operatorname{Im} b_1}$ we get the linear equation with respect to b_m :

$$\begin{aligned} \sum_{j=1}^{m-1} (J^m g)_j \overline{(J^{m-1}g)_j} + (a_{m-1}(J^{m-1}g)_{m-1} + b_m(J^{m-1}g)_m) \overline{(J^{m-1}g)_m} = \\ = \left((\vec{J})^m \vec{g}, (\vec{J})^{m-1} \vec{g} \right). \end{aligned}$$

This equation can be solved since the coefficient $(J^{m-1}g)_m$ is nonzero by (3.3). In order to find a_m we use the relation

$$\|J^m g\|^2 = \left\| (\vec{J})^m \vec{g} \right\|^2$$

which takes the form

$$\begin{aligned} \sum_{j=1}^{m-1} |(J^m g)_j|^2 + |a_{m-1}(J^{m-1}g)_{m-1} + b_m(J^{m-1}g)_m|^2 + \\ + a_m^2 [J^{m-1} \mathbf{g}]_m^2 = \left\| (\vec{J})^m \vec{g} \right\|^2. \end{aligned}$$

Solving for a_m^2 we can find a_m .

According to (5.10) the elements $\{u_{k1}\}_{k=1}^n$ are expressed by means of z_1, z_2, \dots, z_n . The equality $UJg = \vec{J}\vec{g}$ gives the following linear system with respect to the second column $\{u_{k2}\}_{k=1}^n$ of the matrix U :

$$\begin{cases} u_{11}(Jg)_1 + u_{12}(Jg)_2 = (\vec{J}\vec{g})_1 \\ u_{21}(Jg)_1 + u_{22}(Jg)_2 = (\vec{J}\vec{g})_2 \\ \dots \quad \dots \\ u_{n1}(Jg)_1 + u_{n2}(Jg)_2 = (\vec{J}\vec{g})_n \end{cases}.$$

Since $(Jg)_2 = a_1 \sqrt{2 \operatorname{Im} b_1} \neq 0$, one can find $\{u_{k2}\}_{k=1}^n$. By induction the equality $UJ^{m-1}g = (\vec{J})^{m-1}\vec{g}$ enables us to find $\{u_{km}\}_{k=1}^n$ for $m \leq n$.

Finally the relation $J = U^{-1}\vec{J}U$ and already known entries $\{a_k\}_{k=1}^{n-1}$ and $\{b_k\}_{k=1}^{n-1}$ allow us to find the entry b_n .

In conclusion we give formulas for the reconstruction of 2×2 dissipative Jacobi matrix J with a rank one imaginary part from its eigenvalues z_1, z_2 , and the corresponding unitary matrix U :

$$J = \begin{pmatrix} \frac{\operatorname{Re} z_1 \operatorname{Im} z_1 + \operatorname{Re} z_2 \operatorname{Im} z_2}{\operatorname{Im} z_1 + \operatorname{Im} z_2} + i(\operatorname{Im} z_1 + \operatorname{Im} z_2) & \sqrt{\left(\left(\frac{\operatorname{Re} z_1 - \operatorname{Re} z_2}{\operatorname{Im} z_1 + \operatorname{Im} z_2} \right)^2 + 1 \right) \operatorname{Im} z_1 \operatorname{Im} z_2} \\ \sqrt{\left(\left(\frac{\operatorname{Re} z_1 - \operatorname{Re} z_2}{\operatorname{Im} z_1 + \operatorname{Im} z_2} \right)^2 + 1 \right) \operatorname{Im} z_1 \operatorname{Im} z_2} & \frac{\operatorname{Re} z_1 \operatorname{Im} z_2 + \operatorname{Re} z_2 \operatorname{Im} z_1}{\operatorname{Im} z_1 + \operatorname{Im} z_2} \end{pmatrix},$$

$$U = \begin{pmatrix} \sqrt{\frac{\operatorname{Im} z_1}{\operatorname{Im} z_1 + \operatorname{Im} z_2}} & \sqrt{\frac{\operatorname{Im} z_2}{\operatorname{Im} z_1 + \operatorname{Im} z_2}} \\ \sqrt{\frac{\operatorname{Im} z_2}{\operatorname{Im} z_1 + \operatorname{Im} z_2}} & -\sqrt{\frac{\operatorname{Im} z_1}{\operatorname{Im} z_1 + \operatorname{Im} z_2}} \end{pmatrix}.$$

Remark 5.4. *The presented algorithm of reconstruction of the unique dissipative non-self-adjoint Jacobi matrix with a rank one imaginary part having given non-real numbers as its eigenvalues allows to recover the set of tri-diagonal matrices with the same non-real eigenvalues.*

5.3. A non-self-adjoint analog of the Hochstadt and Gesztesy–Simon theorems. The next theorem is a non-self-adjoint analog the of Hochstadt [21] and Gesztesy–Simon [15] results (see Theorems 3.8 and 3.9).

Theorem 5.5. *Let J be an $n \times n$ Jacobi matrix with conditions (1.4). Suppose that the eigenvalues $z_1, \dots, z_k \in \mathbb{C}_+$ counting algebraic multiplicity l_1, \dots, l_k are known as well as*

$$b_1, a_1, b_2, \dots, a_{n-r-1}, b_{n-r},$$

where

$$r = l_1 + l_2 + \dots + l_k.$$

Then $a_{n-r}, b_{n-r+1}, \dots, a_{n-1}, b_n$ are uniquely determined.

Proof. The sub-matrix $J_{[1, n-r]}$ is known. Therefore, the m_- -function $m_-(z, n-r+1)$ is known. The unknown Jacobi sub-matrix $J_{[n-r+1, n]}$ is self-adjoint and the corresponding Weyl function $m_+(z, n-r)$ belongs to the Herglotz–Nevanlinna class. By Cramer’s rule we have that

$$m_+(z, n-r) = \frac{L_{r-1}(z)}{L_r(z)},$$

where $L_{r-1}(z)$ and $L_r(z)$ are polynomials with real coefficients of degrees $r-1$ and r , respectively. Let

$$\Phi(z) := a_{n-r}^2 m_+(z, n-r).$$

By Theorem 4.8 we have

$$\Phi^{(p)}(z_j) = \left(\frac{1}{m_-(z, n-r+1)} \right)^{(p)} \Big|_{z=z_j}, \quad j = 1, \dots, k, \quad p = 0, \dots, l_j.$$

Therefore the values

$$\Phi^{(p)}(z_j), \quad j = 1, \dots, k, \quad p = 0, \dots, l_j$$

are known. Since $\Phi(\bar{z}) = \overline{\Phi(z)}$ and $\operatorname{Im} z_j > 0$, the values

$$\Phi^{(p)}(\bar{z}_j) = \overline{\Phi^{(p)}(z_j)}, \quad j = 1, \dots, k, \quad p = 0, \dots, l_j.$$

are also known. Let $\tilde{\Phi}(z)$ be a ratio of two polynomials with real coefficients of degrees $r - 1$ and r such that

$$\tilde{\Phi}^{(p)}(z_j) = \Phi^{(p)}(z_j), \quad \tilde{\Phi}^{(p)}(\bar{z}_j) = \Phi^{(p)}(\bar{z}_j) \quad j = 1, \dots, k, \quad p = 0, \dots, l_j.$$

Then for the function $\Psi(z) = \tilde{\Phi}(z) - \Phi(z)$ we get

$$\Psi^{(p)}(z_j) = \Psi^{(p)}(\bar{z}_j) = 0, \quad j = 1, \dots, k, \quad p = 0, \dots, l_j.$$

It follows that

$$\Psi(z) = \psi(z) \prod_{j=1}^k (z - z_j)^{l_j} (z - \bar{z}_j)^{l_j},$$

where the function $\psi(z)$ has no zeroes at the points $\{z_j\}_{j=1}^k$ and $\{\bar{z}_j\}_{j=1}^k$. The function $\Psi(z)$ is the ratio of two polynomials. The numerator is of degree less or equal $2r - 1$. Since $r = l_1 + \dots + l_k$ we get $\Psi(z) \equiv 0$. So, $\Phi(z)$ is uniquely determined. Since

$$a_{n-r}^2 = - \lim_{z \rightarrow \infty} z \Phi(z),$$

we get the function $m_+(z, n - r)$ which uniquely determines the self-adjoint Jacobi matrix $J_{[n-r+1, n]}$, i.e. the entries $b_{n-r+1}, \dots, a_{n-1}, b_n$ are uniquely determined. \square

5.4. Extension and refinement of the Hochstadt and Gesztesy–Simon uniqueness theorems for tri-diagonal matrices. Let J be the $n \times n$ tri-diagonal matrix of the form (1.1) with the conditions

$$(5.12) \quad \begin{cases} \operatorname{Im} b_1 > 0, b_k = \bar{b}_k, \quad k = 2, \dots, n, \\ a_k > 0, \quad k \neq p, \quad a_p = 0. \end{cases}$$

Recall (see Subsection 3.5) that we consider a 's and b 's as a single sequence $\{c_k\}_{k=1}^{2n-1}$, where $c_{2k-1} = b_k$ and $c_{2k} = a_k$. The next theorem is the extension and refinement of the Hochstadt theorem [21].

Theorem 5.6. *Let J be an $n \times n$ tri-diagonal matrix of the form (1.1) with conditions (5.12). Suppose that p non-real eigenvalues z_1, \dots, z_p counting algebraic multiplicity and distinct real eigenvalues z_{p+1}, \dots, z_n of J are given as well as $c_{p+n+1}, \dots, c_{2n-1}$. Then these data uniquely determine entries c_1, \dots, c_{p+n} .*

Proof. Since $a_p = 0$ the matrix J takes the block form

$$(5.13) \quad J = \begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix},$$

where J_{11} and J_{22} are $p \times p$ and $(n-p) \times (n-p)$ tri-diagonal matrices of the form

$$(5.14) \quad J_{11} = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{p-1} \\ \cdot & \cdot & \cdot & \cdot & 0 & a_{p-1} & b_p \end{pmatrix},$$

$$(5.15) \quad J_{22} = \begin{pmatrix} b_{p+1} & a_{p+1} & 0 & 0 & \cdot & \cdot & \cdot \\ a_{p+1} & b_{p+2} & a_{p+2} & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & 0 & a_{n-1} & b_n \end{pmatrix}.$$

The matrix J_{22} is self-adjoint and therefore it has only real distinct $n-p$ eigenvalues which are also eigenvalues of J . The matrix J_{11} is prime dissipative Jacobi matrix with a rank one imaginary part. It follows that J_{11} as well as J has p non-real eigenvalues counting multiplicity. By Theorem 5.1 the entries $c_1, c_2, \dots, c_{2p-1}$ can be reconstructed by means of the spectral data z_1, \dots, z_p . The distinct real numbers z_p, \dots, z_n are eigenvalues of self-adjoint Jacobi matrix J_{22} . By the Hochstadt theorem the entries $c_{2p+1}, \dots, c_{2p+(n-p)}$ are determined uniquely by z_{p+1}, \dots, z_n and by the known entries $c_{p+n+1}, \dots, c_{2n-1}$. \square

Next is the extension and refinement of the Gesztesy–Simon theorem [15].

Theorem 5.7. *Let J be an $n \times n$ tri-diagonal matrix of the form (1.1) with conditions (5.12). Suppose that p non-real eigenvalues z_1, \dots, z_p counting algebraic multiplicity and distinct real j eigenvalues of J are given as well as entries*

$$c_{2p+j+1}, \dots, c_{2n-1}.$$

Then these data uniquely determine the matrix J .

Proof. From the proof of Theorem 5.6 and conditions (5.12) the tri-diagonal matrix J takes the form (5.13). By Theorem 5.1 the entries $c_1, c_2, \dots, c_{2p-1}$ can be reconstructed uniquely by means of the non-real spectral data z_1, \dots, z_p . Using the Gesztesy–Simon uniqueness Theorem 3.9 the j distinct real eigenvalues and entries $c_{2p+j+1}, \dots, c_{2n-1}$ uniquely determine the matrix J_{22} . Therefore the given data uniquely determines matrix J . \square

Remark 5.8. Consider more general tri-diagonal $n \times n$ dissipative matrix J with $\text{ran } J_I = \text{span } \{\delta_1\}$ of the form

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdot & \cdot & \cdot \\ \bar{a}_1 & b_2 & a_2 & 0 & \cdot & \cdot & \cdot \\ 0 & \bar{a}_2 & b_3 & a_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & 0 & \bar{a}_{n-1} & b_n \end{pmatrix}$$

with the condition

$$\text{Im } b_1 > 0, \quad b_k = \bar{b}_k, \quad k = 2, \dots, n$$

For such matrices the following statements are equivalent:

- (1) the matrix J has p non-real eigenvalues counting their algebraic multiplicity and $p < n$;
- (2) $a_p = 0$ and $a_k \neq 0$ for $k = 1, \dots, p-1$.

Actually, suppose that $a_p = 0$ and all $a_k \neq 0$ for $k = 1, 2, \dots, p-1$. Then J takes the block form (5.13), where J_{11} and J_{22} are $p \times p$ and $(n-p) \times (n-p)$ tri-diagonal matrices given by (5.14) and (5.15). The matrix J_{11} is prime dissipative Jacobi matrix with a rank one imaginary part and J_{22} is self-adjoint. It follows that J has p non-real eigenvalues counting algebraic multiplicity.

Conversely, suppose that the matrix J has p non-real eigenvalues counting algebraic multiplicity and $p < n$. Let J be the corresponding operator in \mathbb{C}^n . Since $\text{ran } J_I = \text{span } \{\delta_1\}$, the subspace

$$\mathcal{H}_1 = \text{span } \{\delta_1, J\delta_1, \dots, J^n\delta_1\}$$

reduces J , $J_{11} = J|_{\mathcal{H}_1}$ is a prime dissipative operator with a rank one imaginary part, $J_{22} = J|_{(\mathbb{C}^n \ominus \mathcal{H}_1)}$ is a self-adjoint operator. Hence J_{11} has only non-real eigenvalues that coincide with non-real eigenvalues of J . It follows that $\dim \mathcal{H}_1 = p$. From (3.3) it follows that $a_k \neq 0$ for $k = 1, \dots, p-1$, and $a_p = 0$.

6. A NON-SELF-ADJOINT ANALOG OF THE STONE THEOREM

The next theorem is a non-self-adjoint analog of M. Stone's result (see Theorem 3.3).

Theorem 6.1. Let \mathcal{H} be separable Hilbert space and let A be a bounded prime dissipative operator with a rank one imaginary part acting in \mathcal{H} . Then there exists an orthonormal basis in \mathcal{H} in which the matrix of the operator A is a bounded dissipative Jacobi matrix with conditions (1.4).

Proof. Let $\dim \mathcal{H} = n$. Because operator A is prime, it has only n non-real eigenvalues $\{z_k\} \subset \mathbb{C}_+$ counting their algebraic multiplicity. By Theorem 2.7 the characteristic function of A takes the form

$$W(z) = \prod_{k=1}^n \frac{z - \bar{z}_k}{z - z_k}.$$

By Theorem 5.1 there exists a Jacobi matrix

$$\begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & 0 & a_{n-1} & b_n \end{pmatrix}$$

with conditions (1.4) whose eigenvalues coincide with $\{z_k\}$ and the corresponding characteristic function coincides with $W(z)$. It follows that A is unitarily equivalent to the operator in \mathbb{C}^n determined by the matrix J .

Let $\dim \mathcal{H} = \infty$ and let $g \in \mathcal{H}$ such that $2A_I h = (h, g)g$, $h \in \mathcal{H}$. Define the characteristic function of A

$$W(z) = 1 - i \left((A - zI)^{-1} g, g \right), \quad z \in \rho(A)$$

It follows that

$$\lim_{z \rightarrow \infty} z(W(z) - 1) = i \|g\|^2.$$

Let $c = \|g\|^2$. Define

$$V(z) = i \frac{W(z) - 1}{W(z) + 1} = \frac{1}{2} \left((A_R - zI)^{-1} g, g \right)$$

and let

$$m(z) = \frac{2}{\|g\|^2} V(z).$$

Then

$$m(z) = \int \frac{d\rho(t)}{t - z},$$

where the spectral measure has bounded support. Then there exists a unique Jacobi matrix

$$H = \begin{pmatrix} b & a_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

with real entries b, b_2, b_3, \dots and positive entries a_1, a_2, \dots such that $m(z) = ((H - zI)^{-1}\delta_1, \delta_1)$. Note that (1.3) holds because H defines a bounded operator in $l_2(\mathbb{N})$. Moreover, the entries of H can be found by means of continued fraction expansion

$$m(z) = \frac{-1}{z - b} + \frac{-a_1^2}{z - b_2} + \frac{-a_2^2}{z - b_3} + \dots + \frac{-a_{n-1}^2}{z - b_n} + \dots.$$

Let

$$\begin{aligned} J &= \begin{pmatrix} b + i\|g\|^2/2 & a_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \\ &= H + i \begin{pmatrix} \|g\|^2/2 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \end{aligned}$$

The characteristic function of J is

$$1 - i\|g\|^2 ((J - zI)^{-1}\delta_1, \delta_1).$$

Since $J_R = H$, and $m(z) = ((H - zI)^{-1}\delta_1, \delta_1)$, we get that the characteristic function of J coincides with $W(z)$. Because matrix J is prime, the operator A is unitarily equivalent to J . Note that the entries of J can be also found using the continued fraction expansion

$$M(z) = \frac{-1}{z - b_1} + \frac{-a_1^2}{z - b_2} + \frac{-a_2^2}{z - b_3} + \dots + \frac{-a_{n-1}^2}{z - b_n} + \dots,$$

where $M(z) = \frac{i}{\beta}(W(z) - 1)$, and $\beta = \lim_{z \rightarrow \infty} (iz(1 - W(z)))$.

□

From Theorem 6.1 it follows that the Jacobi matrices with conditions (1.3) and (1.4) provide new models for bounded prime dissipative operators with rank one imaginary part alternative to Livsic triangular models described in Subsection 2.3.

Remark 6.2. *From Theorems 2.10 and 6.1 it follows that the equivalent statements (iii), (iv), and (v) are also equivalent to the statement: The function $M(z)$ is the Weyl function of some bounded Jacobi matrix with conditions (1.4).*

Remark 6.3. *In the recent paper [14] it is proved that every cyclic self-adjoint operator in a Pontryagin space is unitary equivalent to some generalized Jacobi matrix [26].*

The next theorem gives more information about the diagonal entries of the model Jacobi matrix.

Theorem 6.4. *Let A be a bounded prime dissipative operator with a rank one imaginary part acting in a separable Hilbert space \mathcal{H} . Suppose that the characteristic function $W(z)$ of A possesses the property*

$$(6.1) \quad W(-z) = \frac{1}{W(z)}, \quad |z| > \|A\|.$$

Then the corresponding to A (by Theorem 2.7) bounded Jacobi matrix of the form (3.1) with the conditions (1.4) possesses the property

$$\operatorname{Re} b_1 = b_2 = \cdots = 0.$$

Proof. Since (see (2.10))

$$V(z) = i \frac{W(z) - 1}{W(z) + 1}, \quad z \in \rho(A) \cap \rho(A_R),$$

we have that in a neighborhood of infinity

$$V(-z) = -V(z).$$

Because

$$m_{A_R}(z) = \frac{1}{l} V(z),$$

where l is the positive eigenvalue of A_I , we get that the Weyl function of A_R is odd. Since $m_{J_R}(z) = m_{A_R}(z)$, from Proposition 3.7 it follows that $\operatorname{Re} b_1 = b_2 = \cdots = 0$. \square

Corollary 6.5. *Let A be a bounded prime dissipative operator with a rank one imaginary part acting in a separable Hilbert space \mathcal{H} . Suppose that eigenvalues of A (counting algebraic multiplicity) are symmetric with respect to the imaginary axis and A has a complete system of root subspaces. If J is the corresponding to A bounded Jacobi matrix with conditions (1.4), then J possesses the property $\operatorname{Re} b_1 = b_2 = \cdots = 0$.*

Proof. By Livsic Theorem 2.6 the characteristic function $W(z)$ takes the form

$$W(z) = \prod_{j=1}^N \left(\frac{z - \bar{z}_j}{z - z_j} \right)^{l_j} \prod_{j=1}^N \left(\frac{z + z_j}{z + \bar{z}_j} \right)^{l_j} \prod_{k=1}^M \frac{z + i\eta_k}{z - i\eta_k},$$

where $\{z_j\}_{j=1}^N$, $\{-\bar{z}_j\}_{j=1}^N$ ($\operatorname{Im} z_j > 0$, $\operatorname{Re} z_j < 0$) are distinct eigenvalues of A , $\{l_j\}_{j=1}^N$ are corresponding algebraic multiplicities, $\{i\eta_k\}_{k=1}^M$, $\eta_k > 0$ are not necessarily distinct pure imaginary eigenvalues of A , N and M are both finite when $\dim \mathcal{H} < \infty$, and one of them or both are infinite

when $\dim \mathcal{H} = \infty$, and $M + 2 \sum_{j=1}^N l_j = \dim \mathcal{H}$. Note that if $M = 0$, (i.e. there are no pure imaginary eigenvalues) then in the finite dimensional \mathcal{H} its dimension is even. It is also possible that $N = 0$.

It follows that

$$W(-z) = \frac{1}{W(z)}, \quad |z| > \|A\|.$$

By Theorem 6.4 we obtain that the corresponding Jacobi matrix possesses the property $\operatorname{Re} b_1 = b_2 = \dots = 0$. \square

Now we want to mention some connections of Theorems 5.1, 6.1 and Corollary 6.5 with some result established by K. Veselić in [30] and [31] related to the spectral problems for the operator pencil of the form

$$(6.2) \quad z^2 M + zC + K,$$

where M , C , and K are self-adjoint $n \times n$ matrices with real entries, M and K are positive definite, C is positive semi-definite. In [30] this problem is reduced to the eigenvalue problem for some $2n \times 2n$ matrix A constructed by means of M , C , and K . The inverse spectral problem for (6.2) is considered in [31] for the case

$$(6.3) \quad M = \operatorname{diag}(m_1, m_2, \dots, m_n),$$

$$(6.4) \quad K = \begin{pmatrix} k_1 & -k_1 & 0 & 0 & \cdot & \cdot & \cdot \\ -k_1 & k_1 + k_2 & -k_2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -k_{n-1} \\ \cdot & \cdot & \cdot & \cdot & 0 & -k_{n-1} & k_{n-1} + k_n \end{pmatrix},$$

and

$$(6.5) \quad C = \begin{pmatrix} \gamma & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \end{pmatrix},$$

where $m_1 = 1$, $\{m_j\}$, $\{k_j\}$, and γ are positive numbers. In such situation the following result is established.

Theorem 6.6. [31]. *Prescribe $2n$ eigenvalues from the open left half-plane which are symmetric (together with their algebraic multiplicities)*

with respect to the real axis. Then there exist unique matrices M , K and C of the form (6.3), (6.4), (6.5), respectively, such that

- (i) $m_j > 0$, $k_j > 0$, $m_1 = 1$,
- (ii) prescribed eigenvalues coincide with the spectrum of the operator pencil (6.2).

Note that in [30] and [31] it is supposed that the total number of negative eigenvalues counting their algebraic multiplicities is even.

As it was mentioned by K. Veselić (in private communication) the matrix A is unitarily similar to $2n \times 2n$ tri-diagonal matrix A' of the form

$$A' = \begin{pmatrix} -\gamma & \alpha_1 & 0 & 0 & \cdot & \cdot & \cdot \\ -\alpha_1 & 0 & \beta_1 & 0 & \cdot & \cdot & \cdot \\ 0 & -\beta_1 & 0 & \alpha_2 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_n \\ \cdot & \cdot & \cdot & \cdot & 0 & -\alpha_n & 0 \end{pmatrix},$$

with $\alpha_j = \sqrt{k_j/m_j}$, $j = 1, \dots, n$, $\beta_j = \sqrt{k_j/m_{j+1}}$, $j = 1, \dots, n-1$. The matrix $J' = -iA'$ is a dissipative tri-diagonal matrix with a rank one imaginary part. Clearly, the matrix J' is unitarily similar to $2n \times 2n$ Jacobi matrix

$$J = \begin{pmatrix} i\gamma & \alpha_1 & 0 & 0 & \cdot & \cdot & \cdot \\ \alpha_1 & 0 & \beta_1 & 0 & \cdot & \cdot & \cdot \\ 0 & \beta_1 & 0 & \alpha_2 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_n \\ \cdot & \cdot & \cdot & \cdot & 0 & \alpha_n & 0 \end{pmatrix}.$$

It follows that the inverse spectral problem considered by K. Veselić is reduced to the reconstruction problem of a dissipative Jacobi matrix with a rank one imaginary part from its eigenvalues. The symmetry property for the eigenvalues of the pencil (6.2) means that the eigenvalues of J counting algebraic multiplicity are symmetric with respect to the imaginary axis. According to Theorem 5.1 there exists a unique $2n \times 2n$ Jacobi matrix with entries satisfying conditions (1.4) whose eigenvalues counting algebraic multiplicity are prescribed numbers from the open upper half-plane and symmetric with respect to imaginary axis. By Corollary 6.5 the diagonal entries of the reconstructed Jacobi matrix possess the property

$$\operatorname{Re} b_1 = b_2 = \dots = b_{2n} = 0.$$

In conclusion we note that the proof of Theorem 6.6 in [31] differs from our proof of Theorem 5.1 and does not use the Livsic characteristic function.

7. DISSIPATIVE VOLTERRA OPERATORS WITH RANK ONE IMAGINARY PARTS AND THE CORRESPONDING SEMI-INFINITE JACOBI MATRICES

Recall that a compact operator is called Volterra operator if its spectrum consists of one point (zero).

Theorem 7.1. [27] *Let \mathcal{H} be a separable Hilbert space. Let A be a prime dissipative Volterra operator with a rank one imaginary part and let*

$$(7.1) \quad A_I h = l(h, e)e$$

for every h , where $l > 0$, $\|e\| = 1$. Then A is unitarily equivalent to the integration operator

$$(Ff)(x) = 2i \int_x^l f(t)dt$$

in the Hilbert space $\mathcal{L}_2[0, l]$.

In connection with Theorem 6.1 we will find a Jacobi matrix corresponding to Volterra operator.

Theorem 7.2. *Let A be a prime dissipative Volterra operator with a rank one imaginary part and let $A_I h = l(h, e)e$ for every h , where $l > 0$, $\|e\| = 1$. Then A is unitarily equivalent to the operator in $l_2(\mathbb{N})$ determined by the dissipative Jacobi matrix*

$$(7.2) \quad J = \begin{pmatrix} il & \frac{l}{\sqrt{1 \cdot 3}} & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{l}{\sqrt{1 \cdot 3}} & 0 & \frac{l}{\sqrt{3 \cdot 5}} & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \frac{l}{\sqrt{3 \cdot 5}} & 0 & \frac{l}{\sqrt{5 \cdot 7}} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{l}{\sqrt{(2n-3)(2n-1)}} & 0 & \frac{l}{\sqrt{(2n-1)(2n+1)}} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Proof. As is well known [27], [10] that if (7.1) holds, then the characteristic function $W(z) = 1 - 2il((A - zI)^{-1}e, e)$ of A takes the form

$$W(z) = \exp\left(\frac{2il}{z}\right).$$

It follows that

$$V(z) = l \left((A_R - zI)^{-1} e, e \right) = i \frac{W(z) - 1}{W(z) + 1} = -\tan \left(\frac{l}{z} \right).$$

By Proposition 2.2 the vector e is a cyclic for A_R , therefore $m(z) = ((A_R - zI)^{-1} e, e)$ is the Weyl function of A_R . We get

$$m(z) = -\frac{1}{l} \tan \left(\frac{l}{z} \right).$$

We consider for simplicity the case $l = 1$. In this case $m(z) = -\tan \left(\frac{1}{z} \right)$. We have to find a self-adjoint semi-infinite Jacobi matrix whose the Weyl function is $-\tan \left(\frac{1}{z} \right)$. This function has the following continued fraction expansion [24]

$$\tan x = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{7 - \frac{x^2}{9 - \dots - \frac{x^2}{2n+1} - \dots}}}}.$$

Comparing with (3.11) and taking into account that $x = 1/z$ we get

$$b_1 = b_2 = \dots = 0, \quad a_n^2 = \frac{1}{(2n-1)(2n+1)}, \quad n = 1, 2, \dots$$

It follows that the Jacobi matrix with the Weyl function $-\tan \left(\frac{1}{z} \right)$ is the following

$$H_0 = \begin{pmatrix} 0 & \frac{1}{\sqrt{1 \cdot 3}} & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{\sqrt{1 \cdot 3}} & 0 & \frac{1}{\sqrt{3 \cdot 5}} & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \frac{1}{\sqrt{3 \cdot 5}} & 0 & \frac{1}{\sqrt{5 \cdot 7}} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{(2n-3)(2n-1)}} & 0 & \frac{1}{\sqrt{(2n-1)(2n+1)}} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Let

$$J_0 = \begin{pmatrix} i & \frac{1}{\sqrt{1 \cdot 3}} & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{\sqrt{1 \cdot 3}} & 0 & \frac{1}{\sqrt{3 \cdot 5}} & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \frac{1}{\sqrt{3 \cdot 5}} & 0 & \frac{1}{\sqrt{5 \cdot 7}} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{(2n-3)(2n-1)}} & 0 & \frac{1}{\sqrt{(2n-1)(2n+1)}} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Then the characteristic function of J_0 is $\exp \left(\frac{2i}{z} \right)$. So the operator J is unitary equivalent to A with $l = 1$. For arbitrary l the Jacobi matrix $J = lJ_0$ takes the form (7.2) and its characteristic function is $\exp \left(\frac{2il}{z} \right)$. \square

Remark 7.3. The real part \mathcal{F}_R of the integration operator $(Ff)(x) = 2i \int_x^l f(t)dt$ takes the form

$$(\mathcal{F}_R f)(x) = i \left(\int_x^l f(t)dt - \int_0^x f(t)dt \right).$$

Its eigenvalues are

$$\frac{2l}{(2k+1)\pi}, \quad k \in \mathbb{Z}.$$

From Theorem 7.2 it follows that these numbers are eigenvalues of the matrix

$$(7.3) \quad H = \begin{pmatrix} 0 & \frac{l}{\sqrt{1 \cdot 3}} & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{l}{\sqrt{1 \cdot 3}} & 0 & \frac{l}{\sqrt{3 \cdot 5}} & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \frac{l}{\sqrt{3 \cdot 5}} & 0 & \frac{l}{\sqrt{5 \cdot 7}} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Remark 7.4. It is known [1], [29] that there are relations between the moments $\{\gamma_k = \int t^k d\rho(t)\}$, $\gamma_0 = 1$ of a measure $d\rho(t)$ and the entries $\{b_k\}$, $\{a_k\}$ of the corresponding self-adjoint Jacobi matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

More precisely, let

$$h_0 = 1, \quad h_k = \det \begin{pmatrix} \gamma_0 & \gamma_1 & \cdot & \cdot & \cdot & \gamma_{k-1} \\ \gamma_1 & \gamma_2 & \cdot & \cdot & \cdot & \gamma_k \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \gamma_{k-1} & \gamma_k & \cdot & \cdot & \cdot & \gamma_{2k-2} \end{pmatrix}, \quad k \geq 1,$$

$$\tilde{h}_k = \det \begin{pmatrix} \gamma_0 & \gamma_1 & \cdot & \cdot & \cdot & \gamma_{k-2} & \gamma_k \\ \gamma_1 & \gamma_2 & \cdot & \cdot & \cdot & \gamma_{k-1} & \gamma_{k+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \gamma_{k-1} & \gamma_k & \cdot & \cdot & \cdot & \gamma_{2k-3} & \gamma_{2k-1} \end{pmatrix}, \quad k \geq 2.$$

Then

$$a_k = \left(\frac{h_{k-1} h_{k+1}}{h_k^2} \right)^{1/2}, \quad b_1 = \gamma_1, \quad \sum_{j=1}^k b_j = \frac{\tilde{h}_k}{h_k}.$$

Since

$$((H - zI)^{-1}\delta_1, \delta_1) = -\frac{1}{l} \tan\left(\frac{l}{z}\right),$$

where H is given by (7.3), the function $\tan\left(\frac{l}{z}\right)$ has the following Taylor expansion at infinity

$$\tan\left(\frac{l}{z}\right) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} \frac{B_{2n}}{z^{2n-1}},$$

where $\{B_{2n}\}$ are Bernoulli numbers [24], $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42, \dots$, we get that

$$\gamma_{2k} = -\frac{l^{2(k-1)}2^{2k}(2^{2k} - 1)B_{2k}}{(2k)!}, \quad \gamma_{2k-1} = 0, \quad k = 1, 2, \dots$$

The formula for $\{a_k\}$ allows to get some relations between Bernoulli numbers.

Theorem 7.5. *Let J_t be a Jacobi matrix of the form*

$$J_t = \begin{pmatrix} t & \frac{l}{\sqrt{1 \cdot 3}} & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{l}{\sqrt{1 \cdot 3}} & 0 & \frac{l}{\sqrt{3 \cdot 5}} & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \frac{l}{\sqrt{3 \cdot 5}} & 0 & \frac{l}{\sqrt{5 \cdot 7}} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{l}{\sqrt{(2n-3)(2n-1)}} & 0 & \frac{l}{\sqrt{(2n-1)(2n+1)}} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where t is a complex number and l is a positive number. Then the corresponding Jacobi operator for all t but two $t = \pm il$ has a complete system of eigenvectors.

Proof. The matrix J_t has the representation

$$J_t = H + t(\cdot, \delta_1)\delta_1,$$

where H is given by (7.3). Suppose that $\text{Im } t \neq 0$. Then the non-real eigenvalues of J_t are solutions of the equation

$$1 + t((H - zI)^{-1}\delta_1, \delta_1) = 0.$$

Because

$$((H - zI)^{-1}\delta_1, \delta_1) = -\frac{1}{l} \tan\left(\frac{l}{z}\right),$$

we get the equation

$$\tan\left(\frac{l}{z}\right) = \frac{l}{t}.$$

This equation has the following solutions:

$$z_k = \frac{2l \left(\arg \left(\frac{t+il}{t-il} \right) + 2\pi k + i \ln \left| \frac{t+il}{t-il} \right| \right)}{\left(\arg \left(\frac{t+il}{t-il} \right) + 2\pi k \right)^2 + \ln^2 \left| \frac{t+il}{t-il} \right|}, \quad k \in \mathbb{Z}, \quad t \neq \pm il.$$

Let

$$x = \ln \left| \frac{t+il}{t-il} \right|, \quad y = \arg \left(\frac{t+il}{t-il} \right), \quad t \neq \pm il.$$

Then

$$\sum_{k=-\infty}^{\infty} \operatorname{Im} z_k = l \sum_{k=-\infty}^{\infty} \frac{2x}{x^2 + (y + 2\pi k)^2}.$$

It is well known that

$$\sum_{k=-\infty}^{\infty} \frac{2x}{x^2 + (y + 2\pi k)^2} = -\frac{\sinh x}{\cos y - \cosh x}.$$

We have

$$\begin{aligned} \sinh x &= \frac{|t+il|^2 - |t-il|^2}{2|t-il||t+il|}, \\ \cos y &= \cos \left(\arg \frac{t+il}{t-il} \right) = \frac{\operatorname{Re} \frac{t+il}{t-il}}{\left| \frac{t+il}{t-il} \right|} = \operatorname{Re} \frac{(t+il)(\bar{t}+il)}{|t-il||t+il|}, \\ \cosh x &= \frac{|t+il|^2 + |t-il|^2}{2|t-il||t+il|}, \\ -\frac{\sinh x}{\cos y - \cosh x} &= -\frac{|t+il|^2 - |t-il|^2}{2\operatorname{Re}((t+il)(\bar{t}+il)) - |t+il|^2 - |t-il|^2} = \\ &= \frac{|t+il|^2 - |t-il|^2}{4l^2} = \frac{\operatorname{Im} t}{l}. \end{aligned}$$

Hence

$$\sum_{k=-\infty}^{\infty} \operatorname{Im} z_k = \operatorname{Im} t.$$

Since in this case the eigenvalues of J_t have the algebraic multiplicities one, according to the Livsic Theorem 2.3 the operator J_t has complete system of eigenvectors.

If $\operatorname{Im} t = 0$, then the operator J_t is self-adjoint and compact, its eigenvalues are real numbers

$$z_k = \frac{2l}{\arg \left(\frac{t + il}{t - il} \right) + 2\pi k}, \quad k \in \mathbb{Z},$$

and the corresponding eigenvectors form a complete system. \square

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DEPARTMENT OF MATHEMATICAL ANALYSIS, EAST UKRAINIAN NATIONAL UNIVERSITY, KVARTAL MOLODYOZHNY 20-A, LUGANSK 91034, UKRAINE
E-mail address: yma@snu.edu.ua

DEPARTMENT OF MATHEMATICS, P.O. BOX 2044, NIAGARA UNIVERSITY, NY 14109, USA
E-mail address: tsekanov@niagara.edu